

The (p, q) -arithmetic hyperbolic lattices; $p, q \geq 6$

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Abstract

We prove there are exactly 16 arithmetic lattices of hyperbolic 3-space which are generated by two elements of finite orders p and q with $p, q \geq 6$. We also verify a conjecture of H.M. Hilden, M.T. Lozano, and J.M. Montesinos concerning the orders of the singular sets of arithmetic orbifold Dehn surgeries on two bridge knot and link complements.

1 Introduction

There are infinitely many lattices in the group $\text{Isom}^+ \mathbb{H}^3 \cong \text{PSL}(2, \mathbb{C})$, of orientation-preserving isometries of hyperbolic 3-space (equivalently Kleinian groups of finite co-volume) which can be generated by two elements of finite orders p and q . For instance, all but finitely many $(p, 0) - (q, 0)$ Dehn surgeries on any of the infinitely many hyperbolic two-bridge links will have fundamental groups which are such uniform (co-compact) lattices [58]. Two infinite families of such groups are shown below in Figure 1.

In [33], we showed that, up to conjugacy, only finitely many of these lattices can be arithmetic. In [34], we identified the 20 such non-uniform lattices of which 15 were *generalised triangle groups*; that is, groups with a presentation of the form $\langle x, y : x^p = y^q = w(x, y)^r = 1 \rangle$ where $w(x, y)$ is a word involving both x and y (see [15, 2]) and $p, q, r \geq 2$.

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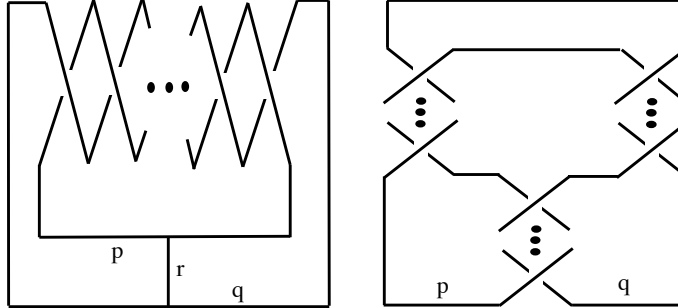


Figure 1. On the left the singular set in \mathbb{S}^3 of an infinite family of hyperbolic generalised triangle groups (eg. $r = 2$ and $p, q \geq 3$) with $p = q$ if there are an odd number of twists, [27]. On the right an infinite family of hyperbolic two bridge p, q links [59].

In this paper we prove that, up to conjugacy, there are exactly 16 arithmetic lattices in $\text{Isom}^+ \mathbb{H}^3$ which can be generated by two elements of finite orders p and q with $6 \leq p, q$. Among these groups there are, curiously, no generalised triangle groups. Two of the groups are non-uniform (and are discussed in [34]) and the others are identified as fundamental groups of orbifolds obtained by Dehn surgeries on 2-bridge knots and links. As such they appear in [25] and our results establish a conjecture in that paper on the degree of the singular set of such orbifolds.

As a basic reference to the deep relationships between arithmetic and hyperbolic geometry we refer to [37]. We note here a few connections. In $\text{Isom}^+ \mathbb{H}^2$, Takeuchi [56] identified all 82 arithmetic lattices generated by two elements of finite order (equivalently arithmetic Fuchsian triangle groups), and, all arithmetic Fuchsian groups with two generators have been identified (see [57, 40]). The connections between arithmetic surfaces, number theory and theoretical physics can be found in work of Sarnak and co-authors eg. [53, 52], see also [29]. In [61] Vinberg gave criteria for Coxeter groups in $\text{Isom} \mathbb{H}^n$ to be arithmetic. Such groups do not exist in the co-compact case for dimension $n \geq 30$, [62]. It has now been established that there are finitely many conjugacy classes of maximal arithmetic Coxeter groups in all dimensions. There are two proofs see [1] and [46] (previously established in two-dimensions in [32]).

Returning to dimension 3, the orientation-preserving subgroups of Cox-

eter groups for tetrahedra, some of which are generated by two elements of finite order, which are arithmetic, are identified in [61] (see [39, 10] for related results). Reid [49] identified the figure eight knot complement as the only arithmetic knot complement. The four orientable hyperbolic 3-manifolds with fundamental group generated by a pair of parabolic elements which are arithmetic are two bridge knot and link complements [21]. The 14 finite co-volume Kleinian groups with two generators, one of finite order, one parabolic, which are arithmetic are described in [11]. An algorithmic approach to deciding if an orbifold obtained by $(p, 0) - (q, 0)$ surgery on a two bridge link (or knot) has arithmetic fundamental groups was given in [25]. Arithmetic hyperbolic torus bundles are discussed in [24, 4] and generalised triangle groups which are Kleinian in [23, 26, 63]. Various extremal groups have been identified as two-generator arithmetic; for instance the minimal volume non-compact hyperbolic 3-manifold and orbifold [43, 6]. The Week's manifold is arithmetic, two-generator and conjecturally the minimal volume orientable hyperbolic 3-manifold [8]. The prime candidate for the minimal volume orientable hyperbolic 3-orbifold is also arithmetic and generated by two elements of finite order [7, 19, 20, 41].

Before precisely stating our main result let us say a few words about its proof and why we have the restriction $p, q \geq 6$. In our work identifying the two-generator non-uniform lattices in [34], a key observation was that the non-compactness hypothesis provided *a priori* knowledge that the underlying fields were quadratic imaginary and the groups we were looking for were commensurable with Bianchi groups. In this paper a major part of the work is to identify the underlying fields. For this we make use of some important results of Stark [55] and Odlyzko [47], as well as results concerning the discriminants of number fields of small degree such as those of Diaz Y Diaz and Olivier [13, 9, 14]. Using these bounds and some results from the geometry of numbers and various discreteness criteria, we bound the degree of the fields in question and then, in turn, bound the possible parameters for an arithmetic Kleinian group - once we have fixed the orders of the generators, the space of all discrete groups up to conjugacy is parameterised by a one complex-dimensional space.

Finally to identify all the groups we use a computer search to examine all algebraic integers in the field satisfying the given bounds and additional arithmetic restrictions on the real embeddings. This procedure gives us a relatively short list of candidate discrete groups which are now known to be

subgroups of arithmetic Kleinian groups [22]. We then use various ideas, discussed in the body of the text, to decide if these groups are in fact arithmetic - at issue here is the finiteness of the co-volume.

Thus we are able identify (up to conjugacy) all the arithmetic Kleinian groups $\langle f, g \rangle$ generated by an element f of order p and g of order q with p and q at least 6. Our results here also give the cases $p = 2$, $q \geq 6$ by the known result that a $(2, p)$ -arithmetic hyperbolic lattice contains a (p, p) -arithmetic hyperbolic lattice with index at most two.

At present the remaining cases $p = 2, 3, 4, 5$ and $q \geq p$ seem computationally infeasible, unless q is large enough - although we have made some recent progress using the work of [16] on the most difficult case $p = 2$ and $q = 3$. As the reader will come to realise, the main problem here is finding effective bounds on the degree of the associated number fields.

Here is our main result:

Theorem 1.1 *Let $\Gamma = \langle f, g \rangle$ be an arithmetic Kleinian group generated by elements of order p and q with $p, q \geq 6$. Then p, q fall into one of the following 4 cases:*

1. $p = q = 6$,
there are precisely 12 groups enumerated below in Table 1. (See comments following the table).
2. $p = q = 8$,
there is precisely one group obtained by $(8, 0)$ surgery on the knot $5/3$.
3. $p = q = 10$,
there is precisely one group obtained by $(10, 0)$ surgery on each component of the link $13/5$.
4. $p = q = 12$,
there are precisely two groups obtained by $(12, 0)$ surgery on the knot $5/3$ and on each component of $8/3$.

Here r/s denotes the slope (or Schubert normal form) of a two bridge knot or link, [5] §12. Thus $5/3$ denotes the well known figure eight knot complement.

As a corollary we are able to verify a condition noticed by Hilden, Lozano and Montesinos [25] concerning the $(n, 0)$ surgeries on two bridge link complements.

Corollary 1.2 *Let $(r/s, n)$ denote the arithmetic hyperbolic orbifold whose underlying space is the 3-sphere and whose singular set is the 2 bridge knot or link with slope r/s and has degree n . Then*

$$n \in \{2, 3, 4, 5, 6, 8, 10, 12, \infty\} \quad (1)$$

We also have the following, slightly surprising, corollary.

Corollary 1.3 *There are no co-compact arithmetic generalised triangle groups with generators of orders at least 6.*

To prove the corollary we shall show later (see 10.1) that there cannot be another presentation of the same group on two generators of orders at least 6 as a generalised triangle group.

In two dimensions, there are in fact 32 arithmetic triangle groups with two generators of orders at least 6 on Takeuchi’s list [56]. In three dimensions for each p and q ($\min\{p, q\} \geq 3$) there are infinitely many co-compact generalised triangle groups with a presentation of the form $\langle f, g : f^p = g^q = w(f, g)^2 = 1 \rangle$ for certain words w in f and g . Some of these are discussed in [27]. Apparently as soon as $\min\{p, q\} \geq 6$ none of these groups can be arithmetic.

Notes: Table 1, and the results above, were produced as follows. The methods we outlined above and discuss in detail in the body of the paper produce for us all possible values of the trace of the commutator of a pair of primitive elliptic generators of an arithmetic Kleinian group (the parameters) as well as an approximate volume for the orbit space. We then use Jeff Weeks’ hyperbolic geometry package “Snappea” [64] to try and identify the orbifold in question by surgering various two bridge knots and links and comparing volumes. Once we have a likely candidate, we use the matrix presentation given by Snappea and verify that the commutator traces are the same. As these traces come as the roots of a monic polynomial with integer coefficients of modest degree, this comparison is exact. Since this trace determines the group up to conjugacy, we thereby identify the orbit space. Conversely, once the two bridge knot or link and the relevant surgery is determined, a value of γ can be recovered from the algorithm in [25].

Next, if α is a complex root of the given polynomial in Table 1, then $k\Gamma = \mathbb{Q}(\alpha)$ and $\alpha(\alpha + 1) = \gamma$. Recall that the parameter γ is determined by the Nielsen equivalence class of a pair of generators of the groups. In

Table 1: Arithmetic groups with $p, q = 6$

γ value	$k\Gamma$	description of orbifold
$i\sqrt{3}$	$z^2 - z + 1$	Γ_{21}
$-1 + i$	$z^2 + 1$	(6,0) surgery on 5/3
-1	$z^2 + z + 1$	Γ_{20}
$1 + 3i$	$z^2 - 2z + 2$	(6,0)-(6,0) surgery on link 24/7
$-1 + i\sqrt{7}$	$z^2 - z + 2$	(6,0)-(6,0) surgery on link 30/11
$-2 + i\sqrt{2}$	$z^2 + 2$	(6,0)-(6,0) surgery on link 12/5
$4.1096 - i \ 2.4317$	$1 + 2z - 3z^2 + z^3$	(6,0) surgery on knot 65/51
$3.0674 - i \ 2.3277$	$2 - 2z^2 + z^3$	(6,0) surgery on knot 13/3*
$2.1244 - i \ 2.7466$	$1 + z - 2z^2 + z^3$	(6,0) surgery on knot 15/11
$1.0925 - i \ 2.052$	$1 - z^2 + z^3$	(6,0) surgery on knot 7/3*
$0.1240 - i \ 2.8365$	$1 + z - z^2 + z^3$	(6,0) surgery on knot 13/3*
$-0.8916 - i \ 1.9540$	$1 + z + z^3$	(6,0)-(6,0) surgery on link 8/5
$-1.8774 - i \ 0.7448$	$1 + 2z + z^2 + z^3$	(6,0) surgery on knot 7/3*
$-2.8846 - i \ 0.5897$	$1 + 3z + z^2 + z^3$	(6,0)-(6,0) surgery on link 20/9

these tables an * denotes that a Nielsen inequivalent pair of generators of order 6 (also listed in the table) gives rise to the same group. Curiously these examples were identified as follows. Each is an index two subgroup of a group obtained by (2,0)-surgery on one component C_1 and (6,0)-surgery on the other component C_2 of a two bridge link, in particular the links 7_1^2 and 9_1^2 in Rolfsen's tables [51]. Suppose images of the meridians are f of order six and g of order two. Then f and gfg^{-1} are generators, both of order six for the group in question. If however we do (6,0)-surgery on C_1 and (2,0)-surgery on C_2 , with images of meridians being f' and g' , then f' and $g'f'g'^{-1}$ give the same group, but are not Nielsen equivalent (as the γ parameters are different). Of course once identified, one can use the retriangulation procedure on Snappea to try to generate these different Nielsen classes of generators (knowing they exist is a big incentive to retriangulate a few times).

The groups listed as Γ_{20} and Γ_{21} are the only non-compact examples and were found in [34]. They have the following presentations

$$\Gamma_{20} = \langle x, y : x^6 = y^6 = [x, y]^3 = ([x, y]x)^2 = (y^{-1}[x, y])^2 = (y^{-1}[x, y]x)^2 = 1 \rangle$$

$$\Gamma_{21} = \langle x, y : x^6 = y^6 = (y^{-1}x)^2 y[x^{-1}, y][x, y][x, y^{-1}]x^{-1} = ([y^{-1}, x]yx^2)^2 = 1 \rangle$$

2 Two-generator Arithmetic Lattices

The group of orientation preserving isometries of the upper half-space model of \mathbb{H}^3 , 3-dimensional hyperbolic space, is given by the group $\mathrm{PSL}(2, \mathbb{C})$, the natural action of its elements by linear fractional transformations on $\hat{\mathbb{C}}$ extending to $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}^+$ and preserving the metric of constant negative curvature via the Poincaré extension.

A subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is said to be *reducible* if all elements have a common fixed point in their action on $\hat{\mathbb{C}}$ and Γ is otherwise *irreducible*. Also Γ is said to be *elementary* if it has a finite orbit in its action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ and Γ is otherwise *non-elementary*.

2.1 Parameters

If $f \in \mathrm{PSL}(2, \mathbb{C})$ is represented by a matrix $A \in \mathrm{SL}(2, \mathbb{C})$ then the trace of f , $\mathrm{tr}(f)$, is only defined up to a sign. However, if $[f, g] = fgf^{-1}g^{-1}$ denotes the commutator of f and g , then $\mathrm{tr}[f, g]$ is well-defined and, furthermore, the two generator group $\langle f, g \rangle$ is reducible if and only if $\mathrm{tr}[f, g] = 2$. For a two-generator group $\langle f, g \rangle$ the three complex numbers $(\gamma(f, g), \beta(f), \beta(g))$

$$\beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4, \quad \gamma(f, g) = \mathrm{tr}[f, g] - 2 \quad (2)$$

are well-defined by f, g and form the *parameters* of the group $\langle f, g \rangle$. They define $\langle f, g \rangle$ uniquely up to conjugacy provided $\langle f, g \rangle$ is irreducible, that is $\gamma(f, g) \neq 0$, see [17].

Now suppose that f and g have finite orders p and q respectively where we can assume that $p \geq q$. In considering the group $\Gamma = \langle f, g \rangle$ we can assume that f and g are primitive elements and so Γ has parameters

$$(\gamma, -4 \sin^2 \pi/p, -4 \sin^2 \pi/q). \quad (3)$$

(Where there is no danger of confusion, we will abbreviate $\gamma(f, g)$ simply to γ .) For fixed p, q , any $\gamma \in \mathbb{C} \setminus \{0\}$ uniquely determines the conjugacy class of such a group $\Gamma = \langle f, g \rangle$. We say Γ is *Kleinian* if it is a discrete non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$. For fixed p and q it is an elementary consequence of a theorem of Jørgensen [28] that the set of all such γ is closed and computer generated pictures suggest that it is highly fractal in nature - for instance the Riley slice, corresponding to two parabolic generators, would correspond to $p = q = \infty$.

The cases where γ is real have been investigated in [30, 31, 35] We have shown in [33], that for each pair (p, q) there are only finitely many γ in \mathbb{C} which yield arithmetic Kleinian groups and for all but a finite number of pairs (p, q) , that finite number is zero. It is our aim here to determine all γ such that Γ is an arithmetic Kleinian group (i.e. a 3-dimensional arithmetic hyperbolic lattice) with $p, q \geq 6$ and to obtain a geometric description of these groups.

2.2 Arithmetic Kleinian Groups

For detailed information on arithmetic Kleinian groups see [3, 60, 37]. For completeness, and since we will rely heavily on these results, we recall here some basic facts.

Let k be a number field and for each place ν of k , let k_ν denote the completion of k with respect to the metric on k induced by the valuation ν . For each Galois monomorphism $\sigma : k \rightarrow \mathbb{C}$, there is an Archimedean valuation given by $|\sigma(x)|$ and if $\sigma(k) \subset \mathbb{R}$ then $k_\nu \cong \mathbb{R}$ and, if not, each complex conjugate pair forms a place and $k_\nu \cong \mathbb{C}$. The other valuations are \mathcal{P} -adic and correspond to prime ideals \mathcal{P} of R_k . The fields $k_\nu = k_{\mathcal{P}}$ are finite extensions of the p -adic numbers \mathbb{Q}_p . Let A be a quaternion algebra over k and let $A_\nu = A \otimes_k k_\nu$ so that A_ν is a quaternion algebra over the local field k_ν . For $k_\nu \cong \mathbb{C}$, then $A_\nu \cong M_2(\mathbb{C})$ but for all other places there are just two quaternion algebras over each local field one of which is $M_2(k_\nu)$ and the other is a unique quaternion division algebra over k_ν .

We say that A is *ramified* at ν if A_ν is a division algebra. The set of places at which A is ramified is finite of even cardinality and is called the *ramification set* of A , denoted by $\text{Ram}(A)$. The ramification set determines the isomorphism class of A over k . We also denote the set of Archimedean ramified places by $\text{Ram}_\infty(A)$ and the non-Archimedean or finite places at which A is ramified by $\text{Ram}_f(A)$. Now as a quaternion algebra A has a basis of the form $1, i, j, ij$ where $i^2 = a, j^2 = b$ and $ij = -ji$, with $a, b \in k^*$. It can thus be represented by a *Hilbert symbol* $\left(\frac{a, b}{k}\right)$. If the real place ν corresponds to the embedding $\sigma : k \rightarrow \mathbb{R}$, then

$$A_\nu \cong \left(\frac{a, b}{k}\right) \otimes_k k_\nu \cong \left(\frac{\sigma(a), \sigma(b)}{\mathbb{R}}\right)$$

and A will be ramified at ν if and only if A_ν is isomorphic to Hamilton's

quaternions. This occurs precisely when both $\sigma(a)$ and $\sigma(b)$ are negative.

Now assume that k has exactly one complex place and that the quaternion algebra A is ramified at least at all the real places of k . Let \mathcal{O} be an order in A and let \mathcal{O}^1 denote the elements of norm 1. In these circumstances there is a k -embedding $\rho : A \rightarrow M_2(\mathbb{C})$ and the group $P\rho(\mathcal{O}^1)$ is a Kleinian group of finite co-volume. The set of *arithmetic Kleinian groups* is the set of Kleinian groups which are commensurable with some such $P\rho(\mathcal{O}^1)$.

It is our aim to identify all (conjugacy classes of) arithmetic Kleinian groups generated by two elements of finite order. To do this we use the identification theorem below which gives a method of identifying arithmetic Kleinian groups from the elements of the given group.

We require the following preliminaries. Let Γ be any non-elementary finitely-generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Let $\Gamma^{(2)} = \langle g^2 \mid g \in \Gamma \rangle$ so that $\Gamma^{(2)}$ is a subgroup of finite index in Γ . Define

$$\left. \begin{aligned} k\Gamma &= \mathbb{Q}(\{\mathrm{tr}(h) \mid h \in \Gamma^{(2)}\}) \\ A\Gamma &= \{\sum a_i h_i \mid a_i \in k\Gamma, h_i \in \Gamma^{(2)}\} \end{aligned} \right\} \quad (4)$$

where, with the usual abuse of notation, we regard elements of Γ as matrices, so that $A\Gamma \subset M_2(\mathbb{C})$.

Then $A\Gamma$ is a quaternion algebra over $k\Gamma$ and the pair $(k\Gamma, A\Gamma)$ is an invariant of the commensurability class of Γ . If, in addition, Γ is a Kleinian group of finite co-volume then $k\Gamma$ is a number field.

We state the identification theorem as follows:

Theorem 2.1 *Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ which is finitely-generated and non-elementary. Then Γ is an arithmetic Kleinian group if and only if the following conditions all hold:*

1. $k\Gamma$ is a number field with exactly one complex place,
2. for every $g \in \Gamma$, $\mathrm{tr}(g)$ is an algebraic integer,
3. $A\Gamma$ is ramified at all real places of $k\Gamma$.
4. Γ has finite co-volume.

It should be noted that the first three conditions together imply that Γ is Kleinian, and without the fourth condition, are sufficient to imply that Γ is a subgroup of an arithmetic Kleinian group.

The first two conditions clearly depend on the traces of the elements of Γ . In addition, we may also find a Hilbert symbol for $A\Gamma$ in terms of the traces of elements of Γ so that the third condition also depends on the traces (for all this, see [38],[37, Chap. 8]).

2.3 Two-generator arithmetic groups

We now suppose that Γ is generated by two elements f, g of orders p and q respectively where $p \geq q$. We have noted that the conjugacy class of Γ is uniquely determined by the single complex parameter γ . We now show how the first three conditions of Theorem 2.1 can be equivalently expressed in terms of γ . This is not true of the fourth condition, but for Γ to have finite co-volume places some necessary conditions on γ (see §3 below).

Note that $\text{tr } f = \pm 2 \cos(\pi/p)$ and $\text{tr } g = \pm 2 \cos(\pi/q)$ are algebraic integers and recall that the traces of all elements in $\langle f, g \rangle$ are integer polynomials in $\text{tr } f, \text{tr } g$ and $\text{tr } fg$. Now the Fricke identity states

$$\gamma = \gamma(f, g) = \text{tr}^2 f + \text{tr}^2 g + \text{tr}^2 fg - \text{tr } f \text{tr } g \text{tr } fg - 4. \quad (5)$$

Thus $\text{tr } fg$ is an algebraic integer if and only if γ is an algebraic integer so that the second condition of Theorem 2.1 is equivalent, in these two-generator cases, to requiring that γ be an algebraic integer.

Now suppose that $p, q \geq 3$. Throughout, we denote $\beta(f), \beta(g)$ (see (2)) by β_1, β_2 respectively so that

$$\beta_1 = -4 \sin^2 \frac{\pi}{p}, \quad \beta_2 = -4 \sin^2 \frac{\pi}{q}, \quad \beta_1 + 4 = 4 \cos^2 \frac{\pi}{p}, \quad \beta_2 + 4 = 4 \cos^2 \frac{\pi}{q}$$

Now $k\Gamma = \mathbb{Q}(\text{tr}^2 f, \text{tr}^2 g, \text{tr } f \text{tr } g \text{tr } fg)$ (see for instance [37, Chap.3]). We consistently use L to denote the totally real subfield

$$L = \mathbb{Q}(\text{tr}^2 f, \text{tr}^2 g) = \mathbb{Q}(\beta_1, \beta_2) = \mathbb{Q}(\cos \frac{2\pi}{p}, \cos \frac{2\pi}{q})$$

Thus $k\Gamma = L(\lambda)$ where $\lambda = \text{tr } f \text{tr } g \text{tr } fg$. From the Fricke identity (5) and $\text{tr}^2(fg) = \lambda^2/(\beta_1+4)(\beta_2+4)$ we deduce that λ satisfies the quadratic equation

$$x^2 - (4 + \beta_1)(4 + \beta_2)x + (4 + \beta_1)(4 + \beta_2)(\beta_1 + \beta_2 + 4 - \gamma) = 0, \quad (6)$$

and that $[k\Gamma : L(\gamma)] \leq 2$.

$$\boxed{\gamma(f, g) \in \mathbb{R}}$$

Let us at this point remove the inconvenient case that $\gamma(f, g)$ is real as this case complicates our discussion. Suppose then that $\gamma \in \mathbb{R}$. In the next section (see (21)), it will be shown that, for Γ to have finite co-volume we must have

$$-4 < \gamma < 4(\cos \pi/p + \cos \pi/q)^2$$

Now if $\gamma \geq 0$, then for any Kleinian group $\Gamma = \langle f, g \rangle$ with $o(f) = p, o(g) = q$ and $\gamma(f, g) = \gamma$, Γ has an invariant plane [30, 35] and so, as the reader can easily verify, cannot have finite co-volume and hence cannot be an arithmetic Kleinian group. Thus $-2 < \text{tr}[f, g] < 2$ and so, whenever Γ is discrete and finite co-volume the commutator $[f, g]$ must be elliptic. All such groups, arithmetic or otherwise, have been determined in [35]. There are precisely nine such groups which are arithmetic, all have $p, q \leq 6$ and there is only one with $p = q = 6$.

Thus we assume henceforth that γ is not real.

Then $k\Gamma$ will be a number field with one complex place if and only if $L(\gamma)$ has one complex place and the quadratic at (6) splits into linear factors over $L(\gamma)$. This implies that, if τ is any real embedding of $L(\gamma)$, then the image of the discriminant of (6), which is $(4 + \beta_1)(4 + \beta_2)(\beta_1\beta_2 + 4\gamma)$, under τ must be positive. Clearly this is equivalent to requiring that

$$\tau(\beta_1\beta_2 + 4\gamma) > 0. \tag{7}$$

Thus $k\Gamma$ has one complex place if and only if (i) $\mathbb{Q}(\gamma)$ has one complex place, (ii) $L \subset \mathbb{Q}(\gamma)$, (iii) for all real embeddings τ of $\mathbb{Q}(\gamma)$, (7) holds and (iv) the quadratic at (6) factorises over $\mathbb{Q}(\gamma)$.

Now, still in the cases where $p, q > 2$, ([37, §3.6])

$$A\Gamma = \left(\frac{\beta_1(\beta_1 + 4), (\beta_1 + 4)(\beta_2 + 4)\gamma}{k\Gamma} \right). \tag{8}$$

Under all the real embeddings of $k\Gamma$, the term $\beta_1(\beta_1 + 4)$ is negative and $(\beta_1 + 4)(\beta_2 + 4)$ is positive. Thus $A\Gamma$ is ramified at all real places of $k\Gamma$ if and only if, under any real embedding τ of $k\Gamma$,

$$\tau(\gamma) < 0. \tag{9}$$

Thus, summarising, we have the following theorem which we will use to determine the possible γ values for the groups we seek.

Theorem 2.2 *Let $\Gamma = \langle f, g \rangle$ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with f of order p and g of order q , $p \geq q \geq 3$. Let $\gamma(f, g) = \gamma \in \mathbb{C} \setminus \mathbb{R}$. Then Γ is an arithmetic Kleinian group if and only if*

1. γ is an algebraic integer,
2. $\mathbb{Q}(\gamma) \supset L = \mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$ and $\mathbb{Q}(\gamma)$ is a number field with exactly one complex place,
3. if $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ such that $\tau|_L = \sigma$, then

$$-\sigma\left(\frac{\beta_1\beta_2}{4}\right) < \tau(\gamma) < 0, \quad (10)$$

4. the quadratic polynomial at (6) factorises over $\mathbb{Q}(\gamma)$,
5. Γ has finite co-volume.

Any non-elementary subgroup $\Gamma = \langle f, g \rangle$ of $\mathrm{PSL}(2, \mathbb{C})$ where $o(f) = o(g) = p > 2$ is contained as a subgroup of index at most 2 in a group $\Gamma^* = \langle h, f \rangle$ where $o(h) = 2$ with

$$\gamma(f, g) = \gamma(h, f)(\gamma(h, f) - \beta_1) \quad (11)$$

and conversely (see [18]). Thus, $(k\Gamma, A\Gamma) = (k\Gamma^*, A\Gamma^*)$, since these are commensurability invariants, and so we can obtain necessary and sufficient conditions for arithmeticity of Γ in terms of $\gamma = \gamma(h, f)$ where $o(h) = 2, o(f) = p > 2$. In this case, $k\Gamma^* = \mathbb{Q}(\mathrm{tr}^2 f, \gamma) = L(\gamma)$ (see [37]) and

$$A\Gamma^* = \left(\frac{\beta_1(\beta_1 + 4), \gamma(\gamma - \beta_1)}{k\Gamma^*} \right). \quad (12)$$

Arguing as above, we have

Theorem 2.3 *Let $\Gamma^* = \langle h, f \rangle$ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with h of order 2 and f of order $p > 2$. Let $\gamma(h, f) = \gamma \in \mathbb{C} \setminus \mathbb{R}$. Then Γ^* is an arithmetic Kleinian group if and only if*

1. γ is an algebraic integer,
2. $\mathbb{Q}(\gamma) \supset L = \mathbb{Q}(\cos 2\pi/p)$ and $\mathbb{Q}(\gamma)$ is a number field with exactly one complex place,

3. if $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ such that $\tau|_L = \sigma$ then

$$\sigma(\beta_1) < \tau(\gamma) < 0, \quad (13)$$

4. Γ^* has finite co-volume.

Implementation of the fourth condition of Theorem 2.2 can be simplified as follows: Suppose that $m(x)$, the minimum polynomial of γ over L , has the form $x^r + a_{r-1}x^{r-1} + \dots + a_0$. From our usual expression for γ at (5), we have:

$$\text{tr}^2 f \text{tr}^2 g \gamma = (\text{tr} f \text{tr} g \text{tr} f g)^2 - \text{tr}^2 f \text{tr}^2 g (\text{tr} f \text{tr} g \text{tr} f g) + \text{tr}^2 f \text{tr}^2 g (\text{tr}^2 f + \text{tr}^2 g - 4). \quad (14)$$

That is $b\gamma = \lambda^2 - b\lambda + c$ where b and c are integers in L . Next, substituting in $m(x)$ and clearing denominators gives $(\lambda^2 - b\lambda + c)^r + a_{r-1}b(\lambda^2 - b\lambda + c)^{r-1} + \dots + a_0b^r = 0$ which is a monic polynomial in λ of degree $2r$ with coefficients integers in L . We define the polynomial

$$M(y) = (y^2 - by + c)^r + a_{r-1}b(y^2 - by + c)^{r-1} + \dots + a_0b^r$$

simply replacing λ by y .

Since $\mathbb{Q}(\lambda) = \mathbb{Q}(\gamma)$, then λ is an algebraic integer in $k\Gamma$ which has a minimum polynomial over L which is monic with integer coefficients. This must also be true of the “other” root $\lambda' = b - \lambda$. So the two factors of the polynomial $M(y)$ have coefficients which are integers in L . Hence the fourth condition of Theorem 2.2 is equivalent to

Lemma 2.4 *The polynomial $M(y)$ factors over L into two monic factors both of degree r and having integral coefficients (in L)*

A slight simplification of this occurs in the cases where $(p, q) > 2$. In these cases, $a = 8 \cos \frac{\pi}{p} \cos \frac{\pi}{q} \cos(\frac{\pi}{p} + \frac{\pi}{q})$ is an algebraic integer in L . If we set $\epsilon = \lambda - a$, then $\mathbb{Q}(\lambda) = \mathbb{Q}(\epsilon)$ and equation (14) takes the form

$$b\gamma = \epsilon(\epsilon - c) \quad (15)$$

where $b = 16 \cos^2 \pi/p \cos^2 \pi/q$, $c = 4 \sin 2\pi/p \sin 2\pi/q$ are integers in L . We can use this factorisation in $m(x)$ to obtain the corresponding result to Lemma 2.4.

See (21) later for an example of this condition applied - using an integral basis for L this condition can be rewritten to assert the existence of a solution in rational integers of a nonlinear system of equations. Since our methods are to deduce the possible minimum polynomials of γ over L , this alternative formulation can be readily computationally implemented. Note that when $p = q$,

$$\epsilon = -4 \cos^2 \frac{\pi}{p} \gamma(h, f)$$

and (11) is a special case of (15) and hence of (14).

3 Free Products

As we have noted, the first four conditions of Theorem 2.2 on γ are sufficient to imply that Γ is a subgroup of an arithmetic Kleinian group. However many of the groups satisfying these four conditions will be isomorphic to the free product $\langle f \rangle * \langle g \rangle$ and so cannot be arithmetic Kleinian groups as they must fail to have finite co-volume. To eliminate these groups we now seek conditions on γ which force a discrete group $\Gamma = \langle f, g \rangle$ to be a free product. Moreover, we will extend the methods of [33] to enumerate the parameters γ which give rise to arithmetic Kleinian groups by obtaining bounds which involve the discriminant of the power basis of $\mathbb{Q}(\gamma)$ over L determined by γ . For this purpose, and also for other methods to be used in the enumeration, we want to obtain as stringent bounds as possible on $|\gamma|, \Im(\gamma), \Re(\gamma)$. The extreme values of these are attained within a contour $\Omega_{p,q}$ in the γ -plane. We thus obtain bounds which are simple functions of one variable which, for each pair (p, q) can be (computationally) maximised.

Define

$$A = \begin{pmatrix} \cos \pi/p & i \sin \pi/p \\ i \sin \pi/p & \cos \pi/p \end{pmatrix}, \quad B = \begin{pmatrix} \cos \pi/q & iw \sin \pi/q \\ iw^{-1} \sin \pi/q & \cos \pi/q \end{pmatrix}.$$

Then if $\Gamma = \langle f, g \rangle$ is a non-elementary Kleinian group with $o(f) = p, o(g) = q$, where $p \geq q \geq 3$ then Γ can be normalised so that f, g are represented by the matrices A, B respectively. The parameter γ is related to w by

$$\gamma = \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{q} \left(w - \frac{1}{w}\right)^2. \quad (16)$$

Given γ , we can further normalise and choose w such that $|w| \leq 1$ and $\operatorname{Re}(w) \geq 0$.

It is convenient here to also consider the cases where $\Gamma^* = \langle h, f \rangle$ with $o(h) = 2, o(f) = p$ as discussed in Theorem 2.3 so that in this section we will allow q to be equal to 2.

We recall the *isometric circles* of a linear fractional transformation

$$g(z) = \frac{az + b}{cz + d} \approx \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C}), \quad c \neq 0$$

are the pair of circles

$$I(g) = \{z : |cz + d| = 1\}, \quad I(g^{-1}) = \{z : |cz - a| = 1\}$$

Notice that $I(g) = \{|g'(z)| = 1\}$ and $I(g^{-1}) = \{|(g^{-1})'(z)| = 1\}$ and that g maps the exterior of $I(g)$ to the interior of $I(g^{-1})$.

The Klein combination theorem, (see [42] for this and important generalisations) can be used to establish the following well known fact: If the isometric circles of g lie inside the intersection of the disks bounded by the isometric circles of f , then $\langle f, g \rangle \cong \langle f \rangle * \langle g \rangle$. (See the illustrative examples in Diagram 1, where this situation holds in case 1 but not in case 2.)

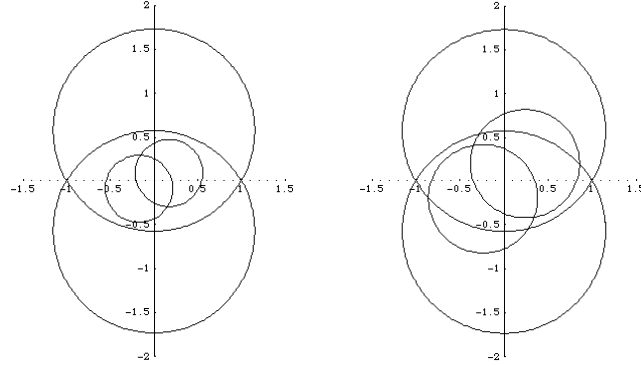


Diagram 1. $p = q = 3$ isometric circles;

1. non-intersecting ($\gamma = -4 + 4i$)
2. intersecting ($\gamma = -1.5 + 1.75i$)

This geometric configuration occurs precisely when

$$|iw \cot \pi/q + i \cot \pi/p| + \frac{|w|}{\sin \pi/q} \leq \frac{1}{\sin \pi/p}. \quad (17)$$

As w traverses the boundary of the region described by (17), then γ traverses a contour $\Omega_{p,q}$, so that, when γ lies outside this, the corresponding group will be a free product. The general shape of such a contour is illustrated by the case exhibited in Diagram 2.

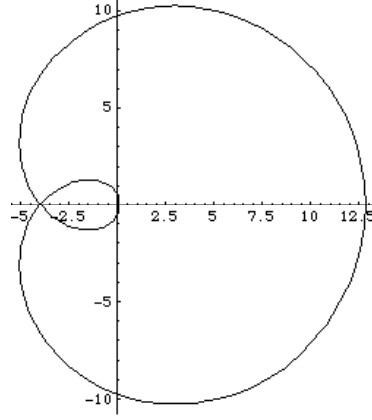


Diagram 2 $\Omega_{8,6}$

More specifically, let $w = re^{i\theta}$, and define $c(p, q) = \cos \pi/p \cos \pi/q$, $s(p, q) = \sin \pi/p \sin \pi/q$. Then on the boundary of the region defined by (17),

$$r^2 + \frac{1}{r^2} = \frac{4(1 + c(p, q) \cos \theta)^2}{s(p, q)^2} - 2 \quad (18)$$

Since

$$\gamma = s(p, q)^2 [(r^2 + r^{-2}) \cos 2\theta - 2 + (r^2 - r^{-2})i \sin 2\theta],$$

and we can assume that $|w| < 1$ and $\Re(w) \geq 0$, we set $\cos \theta = t \in [0, 1]$ and obtain

$$\Omega_{p,q}(t) = \frac{4(2t^2 - 1)(1 + tc(p, q))^2 - 4t^2 s(p, q)^2}{-8t\sqrt{1 - t^2}(1 + tc(p, q))(\sqrt{(1 + tc(p, q))^2 - s(p, q)^2}i)} \quad (19)$$

It is clear that the real part of γ takes its maximum value for $t = 1$ and so

$$\Re(\gamma) \leq 4(\cos \pi/p + \cos \pi/q)^2. \quad (20)$$

and for each (p, q) its minimum value can be computed from this formula. Note that, if γ is real, then

$$-4 \leq \gamma \leq 4(\cos \pi/p + \cos \pi/q)^2. \quad (21)$$

which gives us the estimate we used earlier to handle the case $\gamma \in \mathbb{R}$.

More generally, for $\gamma \in \Omega_{p,q}(t)$, we have

$$|\gamma| = 4[(1 + tc(p, q))^2 - t^2 s(p, q)^2].$$

When $c(p, q) \geq s(p, q)$, which occurs in particular when $p, q \geq 6$,

$$|\gamma| \leq 4(\cos \pi/p + \cos \pi/q)^2. \quad (22)$$

Also, in the case $(p, 2)$, we have $|\gamma| \leq 4$. Finally, note that

$$|\gamma + 4s(p, q)^2| + |\gamma| = 2s(p, q)^2(r^2 + r^{-2}).$$

From the expression for $r^2 + r^{-2}$ at (18) above, this clearly takes its maximal value when $\cos \theta = 1$. Thus if x is a real number in the interval $[-\beta_1\beta_2/4, 0] = [-4s(p, q)^2, 0]$, then

$$|\gamma - x| \leq 4(1 + \cos \pi/p \cos \pi/q)^2. \quad (23)$$

4 The possible values of (p, q)

From Theorem 2.2, we note, first, that γ is an algebraic integer, secondly, that $\mathbb{Q}(\gamma)$ has exactly one complex place and thirdly, that $\mathbb{Q}(\gamma)$ must contain $L = \mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$. Let $[\mathbb{Q}(\gamma) : L] = r$. We now make use of these facts, together with the inequalities that γ and its conjugates must satisfy given in §2 and §3 to produce a list of possible values for the triple (p, q, r) for which there may exist a γ -parameter corresponding to an arithmetic Kleinian group which is not obviously free using the criteria from §3. Further, if (p, q, r) does not appear on this list there cannot be any corresponding arithmetic Kleinian groups (see Table 5). The list obtained in Table 5 is produced by refining a basic list in §4.1 using arguments on the norm and discriminant, each stage being implemented by an elementary program in Maple. The finiteness of such a list was established in [33] and the starting point here uses the crude estimate obtained in [33] that $p, q \leq 120$. In producing our lists, we assume that $p, q \geq 6$ although the methods apply for $p, q \geq 3$.

4.1 Norm method

Let N denote the absolute norm $N : \mathbb{Q}(\gamma) \rightarrow \mathbb{Q}$ and, as before, $L = \mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$. If $(p, q) > 2$, then $L = \mathbb{Q}(\cos 2\pi/M)$ where M is the least common multiple of p and q and otherwise L is of index 2 in that field. Thus if $\mu = [L : \mathbb{Q}]$, then

$$\mu = \begin{cases} \phi(M)/2 & \text{if } (p, q) > 2 \\ \phi(M)/4 & \text{if } (p, q) \mid 2. \end{cases}$$

The field L is totally real and the embeddings $\sigma : L \rightarrow \mathbb{R}$ are defined by

$$\sigma(\cos \frac{2\pi}{p}) = \cos \frac{2\pi j}{p}, \sigma(\cos \frac{2\pi}{q}) = \cos \frac{2\pi j}{q} \text{ where } (j, pq) = 1.$$

Let us denote these embeddings by $\sigma_1, \sigma_2, \dots, \sigma_\mu$, with $\sigma_1 = \text{Id}$. Since γ is an algebraic integer $|N(\gamma)| \geq 1$ and $N(\gamma) = \gamma \bar{\gamma} \prod_{\tau} \tau(\gamma)$ where τ runs over the $r\mu - 2$ real embeddings of $\mathbb{Q}(\gamma)$. If $\tau|_L = \sigma_i$, then by (10)

$$-\frac{\sigma_i(\beta_1\beta_2)}{4} < \tau(\gamma) < 0$$

and from (22), $|\gamma| < 4(\cos \pi/p + \cos \pi/q)^2$. Thus we obtain

$$1 \leq |N(\gamma)| \leq 16(\cos \pi/p + \cos \pi/q)^4 (4s(p, q)^2)^{-2} \prod_{j=1}^{\mu} \left(\frac{\sigma_j(\beta_1\beta_2)}{4} \right)^r. \quad (24)$$

Now letting

$$\delta_n = \begin{cases} 1 & \text{if } n \neq p^\alpha, \text{ } p \text{ a prime} \\ p & \text{if } n = p^\alpha, \text{ } p \text{ a prime,} \end{cases}$$

then, (see [33]), if $\delta_{n,m} = \delta_n^{2/\phi(n)} \delta_m^{2/\phi(m)}$,

$$\prod_{j=1}^{\mu} \sigma_j(\beta_1\beta_2) = \delta_{p,q}^\mu. \quad (25)$$

Thus, taking logs, for a triple (p, q, r) to give rise to a γ which represents an arithmetic Kleinian group it must satisfy the inequality

$$r\mu \leq 4\log \left[\frac{\cos \pi/p + \cos \pi/q}{\sin \pi/p \sin \pi/q} \right] / \log(4/\delta_{p,q}) \quad (26)$$

Note that $r \geq 2$ and $6 \leq p, q \leq 120$ so that we can determine the triples for which (26) holds by obtaining the values of p and q and an upper bound for the related value of r . This produces a list of 86 entries shown in Table 2, which, for future reference, we call the *Norm List*.

p	q	r	p	q	r	p	q	r	p	q	r
6	6	5	7	6	3	7	7	33	8	6	4
8	7	4	8	8	7	9	6	3	9	7	3
9	8	2	9	9	5	10	6	3	10	7	2
10	8	2	10	10	4	11	6	2	11	7	2
11	8	2	11	11	5	12	6	3	12	7	2
12	8	2	12	9	2	12	10	2	12	12	4
13	7	2	13	13	4	14	6	2	14	7	5
14	14	3	15	6	2	15	10	2	15	15	2
16	6	2	16	8	3	16	16	3	17	17	2
18	6	2	18	8	2	18	9	4	18	18	4
19	19	2	20	6	2	20	10	2	20	20	3
21	7	3	21	21	2	22	11	3	22	22	2
23	23	2	24	6	2	24	8	3	24	12	2
24	24	3	26	13	2	26	26	2	28	7	3
28	14	2	28	28	2	30	6	2	30	10	2
30	15	3	30	30	3	32	8	2	32	16	2
32	32	2	34	17	2	36	9	2	36	12	2
36	18	2	36	36	2	38	19	2	40	40	2
42	7	3	42	14	2	42	21	2	42	42	2
44	11	2	48	8	2	48	16	2	48	48	2
54	54	2	60	30	2	60	60	2	66	11	2
70	7	2	84	7	2						

Table 2: Norm List

4.2 Discriminant Method ($r \geq 3$)

This is a refinement of the method used in [33], and we apply it when $r \geq 3$.

If Δ is the discriminant of the power basis $1, \gamma, \gamma^2, \dots, \gamma^{r-1}$ over L and $\delta_{\mathbb{Q}(\gamma)|L}$, the relative discriminant, then

$$|N_{L|\mathbb{Q}}(\Delta)| \geq |N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})|.$$

Choose embeddings $\tau_1, \tau_2, \dots, \tau_\mu$ of $\mathbb{Q}(\gamma)$ into \mathbb{C} such that $\tau_i|_L = \sigma_i$. Then $N_{L|\mathbb{Q}}(\Delta) = \prod_{i=1}^\mu \sigma_i(\Delta)$ and $\sigma_i(\Delta)$ is the discriminant of the power basis $1, \tau_i(\gamma), \tau_i(\gamma^2), \dots, \tau_i(\gamma^{r-1})$ of $\tau_i(\mathbb{Q}(\gamma))$ over L . As in [33], we use Schur's bound [54] which gives that, if $-1 \leq x_1 < x_2 < \dots < x_r \leq 1$ with $r \geq 3$ then

$$\prod_{1 \leq i < j \leq r} (x_i - x_j)^2 \leq M_r = \frac{2^2 3^3 \dots r^r 2^2 3^3 \dots (r-2)^{r-2}}{3^3 5^5 \dots (2r-3)^{2r-3}}. \quad (27)$$

Thus, for $i \geq 2$ we have

$$|\sigma_i(\Delta)| \leq \left(\frac{\sigma_i(\beta_1 \beta_2)}{8} \right)^{r(r-1)} M_r. \quad (28)$$

In the case where $i = 1$, γ has $r - 2$ real conjugates over L denoted by x_3, x_4, \dots, x_r which, by (10), all lie in the interval $(-\beta_1 \beta_2 / 4, 0)$. Thus

$$|\Delta| \leq |\gamma - \bar{\gamma}|^2 \left(\prod_{i=3}^r (\gamma - x_i)^2 (\bar{\gamma} - x_i)^2 \right) \left(\frac{\beta_1 \beta_2}{8} \right)^{(r-2)(r-3)} M_{r-2}. \quad (29)$$

For γ on the contour $\Omega_{p,q}$ we have (see (23))

$$|\gamma - x_i| = |\bar{\gamma} - x_i| < 4(1 + \cos \pi/p \cos \pi/q)^2.$$

We thus define

$$K_1(p, q, r) = 4M_{r-2}[4(1+c(p, q))^2]^{4(r-2)}(2s(p, q)^2)^{(r-2)(r-3)} \text{Max}_{0 \leq t \leq 1} |\Im(\Omega_{p,q}(t))|^2$$

which can be determined using (19).

From (28) and (29) we obtain an upper bound for $|N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})|$. This is bounded below by 1 but since $|N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})| = |\Delta_{\mathbb{Q}(\gamma)}|/\Delta_L^r$, this lower bound may be improved. Since $\mathbb{Q}(\gamma)$ is a field of degree $r\mu$ with exactly one complex place, for $n \geq 2$, let D_n denote the minimum absolute value of the

discriminant of any field of degree n over \mathbb{Q} with exactly one complex place. For small values of n the number D_n has been widely investigated ([9, 13, 14]) and lower bounds for D_n for all n can be computed ([44, 47, 50, 55]). In [48], the bound is given in the form $D_n > A^{n-2}B^2 \exp(-E)$ for varying values of A, B and E . Choosing, by experimentation, suitable values from this table we obtain the bounds shown in Table 3.

Degree n	Bound	Degree n	Bound
2	3	3	27
4	275	5	4511
6	92779	7	2306599
8	68856875*	9	$0.11063894 \times 10^{10}$
10	$0.31503776 \times 10^{11}$	11	$0.90315026 \times 10^{12}$
12	$0.25891511 \times 10^{14}$	13	$0.74225785 \times 10^{15}$
14	$0.21279048 \times 10^{17}$	15	$0.61002775 \times 10^{18}$
16	$0.17488275 \times 10^{20}$	17	$0.50135388 \times 10^{21}$
18	$0.14372813 \times 10^{23}$	19	$0.41203981 \times 10^{24}$
20	$0.11812357 \times 10^{26}$		

Table 3: Discriminant Bounds

* The exact bound in degree 8 is only known for imprimitive fields [9]. This suffices here as the only case not covered here is $p = q = 6$ where, by the Norm List, the degree does not exceed 5.

For any integer $M \geq 2$, let $D(M) = M^{\phi(M)/2} / (\prod_{\pi} \pi^{\phi(M)/(2\pi-2)})$ where the product is over all primes which divide M . Then

$$\Delta_{\mathbb{Q}(\cos 2\pi/M)} = \begin{cases} D(M) & \text{if } M \neq m^\alpha, 2m^\alpha, m \text{ a prime} \\ D(M)/\sqrt{m} & \text{if } M = m^\alpha, 2m^\alpha, m \text{ an odd prime} \\ D(M)/2 & \text{if } M = 2^\alpha, \alpha \geq 2. \end{cases} \quad (30)$$

If $(p, q) > 2$, $L = \mathbb{Q}(\cos 2\pi/M)$ where M is the least common multiple of p and q . If $(p, q) \mid 2$, then $\Delta_L = \Delta_{\mathbb{Q}(\cos 2\pi/p)}^{\phi(q)/2} \Delta_{\mathbb{Q}(\cos 2\pi/q)}^{\phi(p)/2}$.

Thus, from (28) and (29), for all (p, q, r) with $r \geq 3$, the following inequality must hold

$$K_1(p, q, r)(2s(p, q)^2)^{-r(r-1)} (\delta_{p,q}/8)^{\mu r(r-1)} M_r^{\mu-1} \geq \text{Max}\{1, D_{r\mu}/\Delta_L^r\}. \quad (31)$$

Extracting the cases with $r \geq 3$ from the Norm List, and applying this inequality first with a lower bound of 1, results in triples (p, q, r) where the

total degree $r\mu$ is no greater than 20. On these we can apply (31) with values of D_n in Table 3. The result is the so-called *Discriminant List* given in Table 4.

p	q	r	p	q	r	p	q	r
6	6	3,4,5	7	7	3,4,5	8	6	3,4
8	8	3,4,5	9	9	3,4	10	6	3
10	10	3,4	11	11	3	12	6	3
12	12	3,4	14	7	3,4	14	14	3
16	8	3	16	16	3	18	9	3,4
18	18	3,4	20	20	3	24	8	3
24	24	3	30	15	3	30	30	3

Table 4: Discriminant List

4.3 Balancing Method

Once again this is a refinement of an argument used in [33] and here we extend the argument from the case $r = 2$ to all r . Note that the upper bound for $|N(\gamma)|$ used at (24) is attained when the real conjugates of γ cluster at one end of the relevant interval, in which case, the discriminant of the basis using γ will be small. This argument aims to balance these by incorporating both the norm and the discriminant.

Let the minimum polynomial of γ over L have roots $x_1(= \gamma)$, $x_2(= \bar{\gamma})$, x_3, \dots, x_r . Recall that, for each $\tau_i : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ such that $\tau_i|_L = \sigma_i$ we have $\tau_i(\gamma) \in (-\sigma_i(\beta_1\beta_2/4), 0)$. For $i = 2, \dots, \mu$, let

$$\tau_i(x_j) = t_i^{(j)}(-\sigma_i(\beta_1\beta_2/4)), j = 1, 2, \dots, r$$

so that $0 < t_i^{(j)} < 1$.

$$N_{L|\mathbb{Q}}(\text{discr}\{1, \gamma, \gamma^2, \dots, \gamma^{r-1}\}) = (\gamma - \bar{\gamma})^2 \prod_{i=3}^r |\gamma - x_i|^4 \prod_{3 \leq j < k \leq r} (x_j - x_k)^2$$

$$\prod_{i=2}^{\mu} \left(\sigma_i \left(\frac{\beta_1\beta_2}{4} \right)^{r(r-1)} \prod_{1 \leq j < k \leq r} (t_i^{(j)} - t_i^{(k)})^2 \right)$$

where here, and later, all empty products have the value 1. Define

$$R_{p,q} = \prod_{i=2}^{\mu} \left(\sigma_i \left(\frac{\beta_1 \beta_2}{4} \right)^2 \right) = \left(\frac{\delta_{p,q}}{4} \right)^{2\mu} / (4s(p, q)^2)^2.$$

Thus

$$\prod_{i=2}^{\mu} \prod_{1 \leq j < k \leq r} |t_i^{(j)} - t_i^{(k)}| \geq \frac{\text{Max}\{1, (D_{r\mu}/\Delta_L^r)^{1/2}\}}{|\gamma - \bar{\gamma}| \prod_{i=3}^r |\gamma - x_i|^2 \prod_{3 \leq j < k \leq r} |x_j - x_k| R_{p,q}^{r(r-1)/4}}. \quad (32)$$

On the other hand

$$N_{L|\mathbb{Q}}(N_{\mathbb{Q}(\gamma)|L}(\gamma)) = |\gamma|^2 \prod_{i=3}^r x_i \prod_{i=2}^{\mu} \left(-\sigma_i(\beta_1 \beta_2 / 4)^r \prod_{j=1}^r t_i^{(j)} \right)$$

so that

$$\prod_{i=2}^{\mu} \prod_{j=1}^r |t_i^{(j)}| \geq \frac{1}{|\gamma|^2 \prod_{j=3}^r |x_j| R_{p,q}^{r/2}}. \quad (33)$$

Let us define $t_i^{(0)} = 0$ for $i = 2, \dots, \mu$ so that the product of (32) and (33) yields

$$\prod_{i=2}^{\mu} \prod_{0 \leq j < k \leq r} |t_i^{(j)} - t_i^{(k)}| \geq \frac{\text{Max}\{1, (D_{r\mu}/\Delta_L^r)^{1/2}\}}{|\gamma - \bar{\gamma}| |\gamma|^2 \prod_{j=3}^r |\gamma - x_j|^2 \prod_{j=3}^r |x_j| \prod_{3 \leq j < k \leq r} |x_j - x_k| R_{p,q}^{r(r+1)/4}}.$$

Note that $\prod_{0 \leq j < k \leq r} |t_i^{(j)} - t_i^{(k)}| \leq (M_{r+1}/2^{r(r+1)})^{1/2}$. In the same way, for $r > 2$,

$$\prod_{j=3}^r |x_j| \prod_{3 \leq j < k \leq r} |x_j - x_k| \leq \left(M_{r-1} \left(\frac{\beta_1 \beta_2}{8} \right)^{(r-1)(r-2)} \right)^{1/2}.$$

Also with $\gamma \in \Omega_{p,q}$, $|\gamma - \bar{\gamma}| |\gamma|^2 \prod_{j=3}^r |\gamma - x_j|^2$ will be a maximum when all x_j lie at the left hand extremity of the interval $(-\beta_1 \beta_2 / 4, 0)$. So define

$$K_2(p, q, r) = M_{r-1}^{1/2} (2s(p, q)^2)^{(r-1)(r-2)/2} \times$$

$$\text{Max}_{0 \leq t \leq 1} |2\Im(\Omega_{p,q}(t)) \Omega_{p,q}(t)^2 (\Omega_{p,q}(t) + 4s(p, q)^2)^{2(r-2)}|$$

p	q	r	p	q	r	p	q	r
6	6	2,3,4,5	7	6	2	7	7	2,3,4
8	6	2,3	8	8	2,3,4,5	9	6	2
9	9	2,3	10	6	2,3	10	10	2,3
11	11	2	12	6	2,3	12	12	2,3,4
13	13	2	14	7	2,3	14	14	2
15	15	2	16	8	2	16	16	2
18	6	2	18	9	2,3	18	18	2,3
20	10	2	20	20	2	22	11	2
24	8	2	24	12	2	24	24	2
28	7	2	30	10	2	30	15	2
30	30	2,3	36	36	2	42	7	2
42	42	2						

Table 5: Aspiring List

when $r > 2$ and $K_2(p, q, 2) = \text{Max}_{0 \leq t \leq 1} |2\Im(\Omega_{p,q}(t))\Omega_{p,q}(t)^2|$. Thus all our triples (p, q, r) must satisfy

$$K_2(p, q, r) R_{p,q}^{r(r+1)/4} \left(\frac{M_{r+1}}{2^{r(r+1)}} \right)^{(\mu-1)/2} \geq \text{Max}\{1, \left(\frac{D_{r\mu}}{\Delta_L^r} \right)^{1/2}\}. \quad (34)$$

Apply this to the Discriminant List for $r \geq 3$ and to the pairs (p, q) appearing in the Norm List for $r = 2$. In the latter case, if we apply the lower bound of 1 initially, the remaining fields all have total degree not exceeding 20 and we can then utilise Table 3. The end result is shown in Table 5, and termed the *Aspiring List*.

5 Using the field $L = \mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$

From what we have found so far, the Aspiring List, Table 5, has the following property:

If $\gamma \in \mathbb{C} \setminus \mathbb{R}$ is a parameter corresponding to an arithmetic Kleinian group $\Gamma = \langle f, g \rangle$ with f of order p and g of order q and $[\mathbb{Q}(\gamma) : L] = r$, then (p, q, r) must appear on the Aspiring List.

Furthermore, γ will be an algebraic integer which satisfies an irreducible polynomial

$$x^r + c_{r-1}x^{r-1} + \cdots + c_0 = 0 \quad c_j \in R_L. \quad (35)$$

The coefficients c_j are symmetric polynomials in $\gamma, \bar{\gamma}$ and their real conjugates over L . Also the images $\sigma_i(c_j)$ for the real embeddings $\sigma_i : L \rightarrow \mathbb{R}$ are symmetric polynomials in the real conjugates $\tau(\gamma)$ where $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ with $\tau|_L = \sigma_i$, $i \geq 2$.

Thus the bounds on $|\gamma|^2$ and $\Re(\gamma)$ obtained from (19) §3 using the freeness criteria and the bounds on the real conjugates $\tau(\gamma)$ in §2 using the ramification criteria will place bounds on the algebraic integers c_j and $\sigma_i(c_j)$. For each (p, q) we can readily obtain an integral basis for L over \mathbb{Q} . The bounds on γ and its conjugates then translate into bounds on the rational integer coefficients when each c_j is expressed in terms of this integral basis. Once a finite number of possibilities for each coefficient c_j individually is obtained, the roots of each of the resulting finite number of polynomials at (35) so obtained, and their conjugates, can be further examined to see if their roots satisfy the required bounds. We explain the basic methods used to carry out this computational process in this section. This basic method is carried out as a first step by a simple Maple program on the triples in the Aspiring List.

These remarks above actually apply to any algebraic integer δ in $\mathbb{Q}(\gamma)$ such that $\mathbb{Q}(\delta) = \mathbb{Q}(\gamma)$ and for which one can obtain bounds on δ and its conjugates. In particular, if v is a unit in L , we can take $\delta = \gamma/v$ and suitable choices of v lead to improved bounds on δ .

For the basic method which we now describe, we assume first that $\mu \geq 3$, the cases where $\mu \leq 2$ being considerably easier to handle. For all (p, q, r) on the Aspiring List, L has an integral basis of the form $\{1, u, u^2, \dots, u^{\mu-1}\}$ where $u = 2 \cos 2\pi/M$ for some integer M . Let $\sigma_1 = \text{Id}, \sigma_2, \dots, \sigma_\mu$ denote the Galois automorphisms of L over \mathbb{Q} with $\sigma_i(2 \cos 2\pi/M) = 2 \cos 2\pi y_i/M$ where $1 \leq y_i < M/2$ and $(y_i, M) = 1$.

Let δ be an algebraic integer as described above which satisfies (35). Let

$$c_j = m_0 + m_1 u + m_2 u^2 + \cdots + m_{\mu-1} u^{\mu-1}$$

where $m_k \in \mathbb{Z}$. Let A be the $\mu \times \mu$ matrix $[\sigma_i(u^{j-1})]$, $1 \leq i, j \leq \mu$. Then

$$A(\tilde{m}) = \tilde{c}_j \quad (36)$$

where $\tilde{m} = (m_0, m_1, \dots, m_{\mu-1})^t$ and $\tilde{c}_j = (c_j, \sigma_2(c_j), \dots, \sigma_\mu(c_j))^t$. Thus

$$\tilde{m} = A^{-1}\tilde{c}_j \quad (37)$$

where we can numerically determine the entries of A and A^{-1} . The bounds on $|\gamma|^2, \Re(\gamma)$ obtained by maximising them on $\Omega_{p,q}$ using (19) and the bounds on the real conjugates at (10) give bounds on $\delta = \gamma/v, v \in R_L^*$ and its conjugates and hence on each entry of the matrix \tilde{c}_j . Thus there exist $\mu \times 1$ matrices I_j and S_j such that $I_j \leq \tilde{c}_j \leq S_j$ with the obvious notation. In the cases where p is not a prime power, $4\sin^2 \pi/p = -\beta_1$ is a unit and in these cases it is expedient to take $\delta = \gamma/(-\beta_1)$ or $\gamma/(-\beta_2)$ if q is also not a prime power.

Example 5.1 $(p, q, r) = (42, 42, 2)$. In this case with $\delta = \gamma/(-\beta_1)$, $c_0 = |\gamma|^2/(16\sin^4 \pi/42)$ and $0 < \sigma_i(c_0) < \sigma_i(\beta_1\beta_2/4\beta_1)^2 = \sigma_i(\sin^2 \pi/42)^2$. Thus $I_0 < \tilde{c}_0 < S_0$ with $I_0 = 0$, and the i -th entry s_i of S_0 is $\sigma_i(\sin^2 \pi/42)^2$ for $i = 2, 3, \dots, 6$ and $s_1 = 16(\cos^2 \pi/42/\sin^2 \pi/42)^2$.

Remark. From this example, a common feature of many examples will be noted - that all entries of S_0 except the first are small. This is a consequence of our choice of v and we will explain below how to exploit this.

Let us return to the general case as at (37). We can obtain upper and lower estimates on \tilde{m} as follows: Write $A^{-1} = A_+^{-1} + A_-^{-1}$ where A_+^{-1}, A_-^{-1} are $\mu \times \mu$ matrices with all entries in A_+^{-1} being ≥ 0 and those in A_-^{-1} being ≤ 0 . We thus obtain

$$A_+^{-1}I_j + A_-^{-1}S_j \leq \tilde{m} \leq A_+^{-1}S_j + A_-^{-1}I_j. \quad (38)$$

This then gives a finite number of possibilities for \tilde{m} . We refer to this as a search space and from these inequalities, its size can be readily measured. In general, the search space described by (38) can be extremely large. In Example 5.1 above, for example, it is of the order of 1.5×10^{25} . In such cases, we extend this technique to exploit the fact that, in many cases, all the entries of I_j and S_j except the first are small.

From (38) determine the possible values of m_0 , the first entry of \tilde{m} and the constant term in the expression of c_j in terms of the integral basis $1, u, u^2, \dots, u^{\mu-1}$. For each m_0 we have

$$m_1u + m_2u^2 + \dots + m_{\mu-1}u^{\mu-1} = c_j - m_0 \quad (39)$$

and the corresponding $\mu - 1$ equations under the embeddings $\sigma_i, i = 2, \dots, \mu$. Now if B denotes the $\mu - 1 \times \mu - 1$ matrix obtained from A by deleting the first row and first column and if \tilde{m}', \tilde{c}_j' denote the $\mu - 1 \times 1$ matrices obtained by removing the first entries of \tilde{m}, \tilde{c}_j , we can write the $\mu - 1$ equations obtained from (39) for the embeddings $\sigma_2, \dots, \sigma_\mu$, in the form

$$B\tilde{m}' = \tilde{c}_j' - m_0\tilde{1}$$

where $\tilde{1}$ is the $\mu - 1 \times 1$ matrix all of whose entries are 1. This then yields $\tilde{m}' = B^{-1}\tilde{c}_j' - m_0B^{-1}\tilde{1}$. For each m_0 the term $m_0B^{-1}\tilde{1}$ is fixed. By splitting B^{-1} into its positive and negative parts as we did for A^{-1} and using the truncated limits I_j', S_j' for \tilde{c}_j we obtain bounds on \tilde{m}' given by

$$B_+^{-1}I_j' + B_-^{-1}S_j' - m_0B^{-1}\tilde{1} \leq \tilde{m}' \leq B_+^{-1}S_j' + B_-^{-1}I_j' - m_0B^{-1}\tilde{1}. \quad (40)$$

If the entries of I_j', S_j' are small, this yields a small search space for \tilde{m}' whose size is essentially independent of m_0 . In Example 5.1, for example, there are 6166 possibilities for m_0 and 576 for \tilde{m}' so that the search space is now of the order of 3.5×10^6 , a significant reduction.

For each resulting \tilde{m} we check the validity of $I_j \leq A\tilde{m} \leq S_j$ and list the resulting \tilde{m} and hence candidate c_j . Again in Example 5.1, there are three such integer vectors \tilde{m} and hence only three candidates for c_0 .

In the cases where $\mu = 2$, we dispense with the use of the matrix A (and hence B). For in that case, all integers in R_L have the form $(a + b\sqrt{d})/2$ where $a, b \in \mathbb{Z}$ with $a \equiv b \pmod{2}$ and $a \equiv b \equiv 0 \pmod{2}$ if $d \not\equiv 1 \pmod{4}$. Thus if $c_j = (a_j + b_j\sqrt{d})/2$, the upper and lower bounds on c_j and $\sigma(c_j)$ respectively for σ the non-identity embedding, can be expressed as

$$\ell_1 < \frac{a_j + b_j\sqrt{d}}{2} < u_1, \quad \ell_2 < \frac{a_j - b_j\sqrt{d}}{2} < u_2.$$

Thus a_j must be an integer between $\ell_1 + \ell_2$ and $u_1 + u_2$ and, for each such a_j , b_j lies between $(2\ell_1 - a_j)/\sqrt{d}$ and $(2u_1 - a_j)/\sqrt{d}$. Provided the bounds are reasonable, it is a simple matter to find all the integers satisfying these inequalities.

These methods described above for enumerating and listing candidate values of the coefficients c_j in either the cases where $\mu \geq 3$ or $\mu = 2$ will be referred to as the *Basic Method*.

In the (many) cases where $p = q$, we noted in §2 that any non-elementary Kleinian group generated by f, g where $o(f) = o(g) = p$ is a subgroup of index at most 2 in a non-elementary Kleinian group generated by f and an element h of order 2. Thus, in these cases, by Theorem 2.3, instead of trying to determine $\gamma = \gamma(f, g)$, we can search for possible values of $\gamma_1 = \gamma(h, f)$. For a real embedding $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ with $\tau_L = \sigma$ we have, by (13), $-\sigma(4 \sin^2 \pi/p) < \tau(\gamma_1) < 0$. Also from §3, $|\gamma_1| \leq 4$. Furthermore, by (11), $\gamma_2 = \beta_1 - \gamma_1$ also corresponds to a group generated by an element of order 2 and an element of order p . Thus we can assume that the γ_1 -space is symmetric about $\Re(\gamma_1) = \beta_1/2$ and so

$$-4 < \Re(\gamma_1) < -2 \sin^2 \pi/p. \quad (41)$$

We can thus apply the same strategy as in the *Basic Method* to determine the coefficients c_j of the polynomial satisfied by γ_1 or $\delta_1 = \gamma_1/v$ for a suitable unit $v \in R_L$. We refer to this also as a *Basic Method*.

Applying the *Basic Methods* to triples on the Aspiring List yields candidate values for the coefficients c_j of the polynomials $p(x)$ satisfied by some δ where $\mathbb{Q}(\delta) = \mathbb{Q}(\gamma)$. In some cases the bounds are tight enough that there are no candidate values for one of the coefficients. We list these below in Table 6. In this Table and subsequently, we will use the notation $\gamma(p, q)$ for $\gamma(f, g)$ where $o(f) = p, o(g) = q$ and also $\gamma(2, p)$ for $\gamma(h, f)$ where $o(h) = 2$. Generally, the search spaces are small in the cases of coefficients c_0 and c_{r-1} as they are, up to sign, the product and sum of the roots. Thus degree 2, considered in §6 below, is reasonably straightforward. For the other coefficients, additional methods may be required to reduce the size of the search space to manageable proportions. These will be discussed in §8 to 9 below.

Triple	δ	Outcome
(28,7,2)	$\gamma(28/7)/4 \sin^2 \pi/28$	No values of c_0
(22,11,2)	$\gamma(22, 11)$	No values of c_0
(16,8,2)	$\gamma(16, 8)/(1 + 2 \cos 6\pi/16)$	No values of c_1 .

Table 6:

6 Degree 2

Here we consider the cases where $r = 2$ so that δ satisfies $p(x) = x^2 + c_1x + c_0$. From the Basic Methods we have obtained candidate values for c_1 and c_0 . The polynomial $p(x)$ will define a field with one complex place if and only if $c_1^2 - 4c_0 < 0$ and $\sigma_i(c_1^2 - 4c_0) > 0$ for $i = 2, 3, \dots, \mu$. Furthermore, for $i \geq 2$ both roots of $p^{\sigma_i}(x) = x^2 + \sigma_i(c_1)x + \sigma_i(c_0) = 0$ must lie in an interval $(-\ell_i, 0)$ where $\ell_i > 0$ is the bound obtained using (10) or (13) for the particular choice of δ . By the Basic Method, $0 \leq \sigma_i(c_1) < 2\ell_i$ and $0 < \sigma_i(c_0) < \ell_i^2$ for $i \geq 2$. Thus the condition on the location of these real roots is equivalent to requiring that $\ell_i^2 - \sigma_i(c_1)\ell_i + \sigma_i(c_0) > 0$. Thus all these conditions can be checked directly on the candidate coefficients c_1, c_0 . This will be referred to as *polynomial reduction*.

Examples 6.1 (1.) $(p, q, r) = (42, 42, 2)$. As in Example 5.1, take $\delta = \gamma(42, 42)/4 \sin^2 \pi/42$. The Basic Method throws up two candidates for c_1 and three for c_0 . None of the 6 resulting polynomials satisfy all the inequalities above and so there are no arithmetic Kleinian groups corresponding to the triple $(42, 42, 2)$.
 (2.) $(p, q, r) = (24, 24, 2)$. Taking $\delta = \gamma(2, 24)$ the Basic Method yields 74 candidates for c_0 and 20 for c_1 . Then polynomial reduction reduces this to two polynomials.
 (3.) $(p, q, r) = (12, 6, 2)$. With $\delta = \gamma(12, 6)$ we obtain 45 candidates for c_0 and 19 for c_1 and polynomial reduction reduces this to a total of 45 polynomials.

The remaining polynomials can then be computationally solved and the complex roots checked to see if they give rise to values of $\gamma(p, q)$ which lie inside the contour $\Omega_{p,q}$. All the polynomials which are left at this stage correspond to a γ which satisfies conditions 1, 2 and 3 of Theorems 2.2 or 2.3. If $p = q$ and the deduction is carried out using $\gamma(2, p)$, then the resulting $\gamma(p, p) = \gamma(2, p)(\gamma(2, p) + 4 \sin^2 \pi/p)$ corresponds to a subgroup of an arithmetic Kleinian group. It can turn out that the resulting $\gamma(p, p)$ is real, which cases, as noted in §2, are completely understood.

Example 6.2 $(p, q, r) = (24, 24, 2)$. The two polynomials (see above) both yield that $\gamma(24, 24)$ is real and there are no such arithmetic Kleinian groups. (See §2.3).

More generally, we still need to check condition 4 of Theorem 2.2 for $\gamma(p, q)$ using Lemma 2.4. If $p = q$ and $\gamma(p, p)$ is obtained by first determining $\gamma(2, p)$, then this condition is automatically satisfied as noted at the end of §2.3. Thus this is most frequently applied in the cases where $p \neq q$.

Example 6.3 $(p, q, r) = (12, 6, 2)$. Here the field $L = \mathbb{Q}(\sqrt{3})$ and we have 45 candidate polynomials from above. Using (15) we replace the variable x by $y(y - \sqrt{3})/(3(2 + \sqrt{3}))$ and find that just one of the resulting quartic polynomials in y factorise in $\mathbb{Q}(\sqrt{3})$. Thus there is one value of $\gamma(12, 6)$ which gives rise to a subgroup of an arithmetic Kleinian group in this case.

Using this factorisation method any remaining polynomials will give values of γ which correspond to subgroups of arithmetic groups. The results are shown in Table 7. These parameters must then be subjected to geometric methods to ascertain if they have finite covolume and so satisfy the final conditions of Theorems 2.2 or 2.3. These geometric methods will be described in §10.

The notation used in Table 7 is as follows: the second column gives the generating element δ to which we apply the Basic Method. The next two columns give the number of resulting possible values of the coefficients c_0 and c_1 . The column headed “PR”, refers to the number of polynomials remaining after polynomial reduction, that headed “B” gives the number that are non-real and lie inside the contour $\Omega_{p,q}$ and the “F” column those left after the factorisation criteria has been applied. Thus the non-zero entries in the final column are those which need to be further considered by geometric methods. (The * in the $(42, 7, 2)$ row indicates that the values of c_1 were calculated and from the small number of resulting values we obtained improved bounds on c_0 by using the inequalities implied by the method of polynomial reduction. The - in the row of $(14, 7, 2)$ indicates that we omitted this step.)

7 Degree 3

. Apart from the *polynomial reduction* process, this is very similar to the degree 2 cases as carried out in the preceding section. Let δ be such that $\mathbb{Q}(\delta) = \mathbb{Q}(\gamma)$ where $[Q(\gamma) : L] = 3$ so that δ satisfies $p(x) = x^3 + c_2x^2 + c_1x + c_0 = 0$ with $c_i \in R_L$. Using the Basic Methods we obtain candidate values for c_0, c_1 and c_2 . In general, there are many more candidates for c_1 than for c_0 or c_2 . We then ascertain that at the non-identity real places

Triple	δ	c_0	c_1	PR	B	F
(42,42,2)	$\gamma(42, 42)/4 \sin^2 \pi/42$	3	2	0	0	0
(42,7,2)	$\gamma(42, 7)/4 \sin^2 \pi/42 \times 4 \sin^2 \pi/21$	*	5	0	0	0
(36,36,2)	$\gamma(2, 36)$	16	10	0	0	0
(30,30,2)	$\gamma(2, 30)$	249	44	10	1	1
(30,15,2)	$\gamma(30, 15)/4 \sin^2 \pi/15$	36	20	0	0	0
(30,10,2)	$\gamma(30, 10)/4 \sin^2 \pi/10$	9	8	0	0	0
(24,24,2)	$\gamma(2, 24)$	72	20	2	0	0
(24,12,2)	$\gamma(24, 12)/4 \sin^2 \pi/12$	6	5	0	0	0
(24,8,2)	$\gamma(24, 8)$	12	12	1	0	0
(20,20,2)	$\gamma(20, 20)/4 \sin^2 \pi/20$	16	13	0	0	0
(20,10,2)	$\gamma(20, 10)/4 \sin^2 \pi/20$	1	4	0	0	0
(18,18,2)	$\gamma(2, 18)$	122	30	16	3	3
(18,9,2)	$\gamma(18, 9)/4 \sin^2 \pi/18$	268	62	73	47	2
(18,6,2)	$\gamma(18, 6)/4 \sin^2 \pi/18$	6	9	0	0	0
(16,16,2)	$\gamma(2, 16)$	61	19	0	0	0
(15,15,2)	$\gamma(15, 15)/4 \sin^2 \pi/15$	4	5	0	0	0
(14,14,2)	$\gamma(14, 14)/4 \sin^2 \pi/14$	85	38	10	3	0
(14,7,2)	$\gamma(14, 7)/4 \sin^2 \pi/14$	244	65	161	-	1
(13,13,2)	$\gamma(2, 13)$	11	13	0	0	0
(12,12,2)	$\gamma(2, 12)$	64	17	67	18	18
(12,6,2)	$\gamma(12, 6)$	45	19	45	30	1
(11,11,2)	$\gamma(2, 11)$	35	17	0	0	0
(10,10,2)	$\gamma(2, 10)$	48	8	44	15	15
(10,6,2)	$\gamma(10, 6)$	34	20	40	24	0
(9,9,2)	$\gamma(2, 9)$	72	22	7	2	2
(9,6,2)	$\gamma(9, 6)$	4	7	1	0	0
(8,8,2)	$\gamma(2, 8)$	65	17	48	20	20
(8,6,2)	$\gamma(8, 6)$	42	21	50	33	0
(7,7,2)	$\gamma(2, 7)$	199	43	32	8	8
(7,6,2)	$\gamma(7, 6)$	8	12	0	0	0
(6,6,2)	$\gamma(2, 6)$	16	8	78	24	24

Table 7: Degree 2 candidates

σ_i of L , the conjugate polynomials $p^{\sigma_i}(x)$ has three real roots in the interval $(-\ell_i, 0)$ where ℓ_i is obtained from (10) and (13). This can be checked without numerically solving the polynomial (which is a time consuming process) by the following sequence of requirements on combinations of the coefficients:

- $\sigma_i(c_2)^2 > 3\sigma_i(c_1)$, which forces the derivative $Dp^{\sigma_i}(x)$ to have two real roots;
- $Dp^{\sigma_i}(-\ell_i) > 0$, which forces these roots, r_1, r_2 to lie in the interval $(-\ell_i, 0)$;
- $p^{\sigma_i}(-\ell_i) < 0$, which forces $p^{\sigma_i}(x)$ to have at least one root in the interval $(-\ell_i, 0)$;
- $\sigma_i(-2c_2c_1c_0/3 + 4c_1^3/27 + 4c_2^3c_0/27 - c_2^2c_1^2/27 + c_0^2) < 0$. which forces $p^{\sigma_i}(r_1)p^{\sigma_i}(r_2) < 0$ and so $p^{\sigma_i}(x)$ to have three real roots in the interval $(-\ell_i, 0)$.

Any cubic remaining after this, can then be solved at the identity real place of L to ensure that it has a pair of non-real roots and that the real root lies in the interval $(-\ell_1, 0)$. Following this *polynomial reduction* procedure, we check to determine if the values of $\gamma(p, q)$ lie inside the contour $\Omega_{p,q}$. Finally, if appropriate, we apply the factorisation condition of Theorem 2.2. The results are tabulated in Table 8, as in Table 7 so that any non-zero numbers in the right hand column correspond to groups which must be checked by geometric methods to see if they have finite covolume.

Note: The * in case (30, 30, 3) indicates that we actually used the linked triples method which is explained in the next sections. The outcome was that there were no linked triples and so no groups can arise.

8 Degree ≥ 4

From the *Aspiring List*, we see that there are six cases with $r \geq 4$ all with $p = q$. In general, the Basic Methods enable one to determine the candidates for the coefficients c_0 and c_{r-1} , but give rise to unfeasible search spaces in attempting to determine the other coefficients. So we develop some new methods of obtaining bounds on the coefficients by exploiting the relationship at (11) between $\gamma = \gamma(p, p)$ and $\gamma_1 = \gamma(2, p)$. Since $\gamma_2 = \beta_1 - \gamma_1$ is also a

Triple	δ	c_0	c_1	c_2	PR	B	F
(30,30,3)	$\gamma(2, 30/4 \sin^2 \pi/30)$	250	*	296	-	-	0
(18,18,3)	$\gamma(2, 18)/4 \sin^2 \pi/18$	8	4442	180	1	0	0
(18,9,3)	$\gamma(18, 9)/4 \sin^2 \pi/18$	11	2429	137	0	0	0
(14,7,3)	$\gamma(14, 7)/4 \sin^2 \pi/14$	25	2207	148	1	0	0
(12,12,3)	$\gamma(2, 12)$	65	218	26	85	19	19
(12,6,3)	$\gamma(12, 6)$	3	138	30	1	1	1
(10,10,3)	$\gamma(2, 10)$	48	175	29	33	5	5
(10,6,3)	$\gamma(10, 6)$	1	103	32	0	0	0
(9,9,3)	$\gamma(2, 9)$	219	812	56	0	0	0
(8,8,3)	$\gamma(2, 8)$	133	256	30	268	29	29
(8,6,3)	$\gamma(8, 6)$	5	129	32	2	0	0
(7,7,3)	$\gamma(2, 7)$	1381	2449	105	26	1	1
(6,6,3)	$\gamma(2, 6)$	16	24	9	1496	124	124

Table 8: Degree 3 candidates

candidate $\gamma(2, p)$ value, equation (11) can be stated as

$$\gamma = -\gamma_1 \gamma_2. \quad (42)$$

We use the most “awkward” case $(7, 7, 4)$, which is the case of highest total degree over \mathbb{Q} amongst these six, as a template to describe our methods.

Let $\beta_1 = -(2 - 2 \cos 2\pi/7)$ and $\beta_i = \sigma_i(\beta_1)$ where σ_i , $i = 2, 3$ are the non-trivial automorphisms of $L = \mathbb{Q}(\cos 2\pi/7)$. Let $B_i = -\beta_i^2/4$ so that, for $\tau : k\Gamma \rightarrow \mathbb{R}$, we have

$$\beta_i < \tau(\gamma_1), \tau(\gamma_2) < 0 \text{ and } B_i < \tau(\gamma) < 0 \quad (43)$$

where $\tau|_L = \sigma_i$. From §3, we also have bounds on the complex number γ i.e.

$$|\gamma| < 4(2 \cos \pi/7)^2 = G_u \text{ and } R_\ell < \Re(\gamma) < G_u \quad (44)$$

where $R_\ell \approx -5.0914$ computed using (19). Since γ_1, γ_2 are symmetric about $\Re(\gamma_i) = \beta_1/2$, we can assume that $|\gamma_1| < 4$ and $-4 < \Re(\gamma_1) \leq \beta_1/2$ and $|\gamma_2| < 4$ and $\beta_1/2 \leq \Re(\gamma_2) < \beta_1 + 4$ (see §3).

Let $\gamma, \gamma_1, \gamma_2$ satisfy the polynomials

$$\left. \begin{aligned} p(x) &= x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 \\ p_1(x) &= x^4 + c_3^{(1)} x^3 + c_2^{(1)} x^2 + c_1^{(1)} x + c_0^{(1)} \\ p_2(x) &= x^4 + c_3^{(2)} x^3 + c_2^{(2)} x^2 + c_1^{(2)} x + c_0^{(2)} \end{aligned} \right\} \quad (45)$$

respectively. As noted above, we can determine $c_0, c_0^{(1)}, c_0^{(2)}, c_3, c_3^{(1)}, c_3^{(2)}$ by our Basic Methods. The basic ideas here are then to use these determined values to place bounds and restrictions on the remaining coefficients. Furthermore, since $\gamma_2 = \beta_1 - \gamma_1$, the coefficients of $p_2(x)$ are combinations of the coefficients of $p_1(x)$. All this enables us to determine $c_2^{(1)}$ and $c_1^{(1)}$ from the other coefficients.

Using the Basic Methods we determine candidates for c_0 and $c_0^{(1)}$ (and hence $c_0^{(2)}$). There are 412 and 9769 respectively. From (42) it follows that $c_0 = c_0^{(1)} c_0^{(2)}$ and we determine all such linked triples $(c_0, c_0^{(1)}, c_0^{(2)})$. (In this $(7, 7, 4)$ case it is expedient to first narrow down the search by using the fact that the rational integral equation $N_{L|\mathbb{Q}}(c_0) = N_{L|\mathbb{Q}}(c_0^{(1)}) N_{L|\mathbb{Q}}(c_0^{(2)})$ must hold.) There are 8979 linked triples. Since $\gamma_2 = \beta_1 - \gamma_1$, then

$$c_0^{(2)} = \beta_1^4 + c_3^{(1)} \beta_1^3 + c_2^{(1)} \beta_1^2 + c_1^{(1)} \beta_1 + c_0^{(1)}. \quad (46)$$

This implies that $\beta_1 \mid c_0^{(1)} - c_0^{(2)}$, which, if $c_0^{(1)} - c_0^{(2)} = a + bu + cu^2$ where $u = 2 \cos 2\pi/7$ is equivalent to $a + 2b + 4c \equiv 0 \pmod{7}$. We reduce our set of linked triples to satisfy this divisibility condition, obtaining 1303 such triples.

For each candidate linked triple, we now obtain new bounds on the coefficients $c_1, c_1^{(1)}, c_1^{(2)}$ and their conjugates which depend on the values of a linked triple as follows: Let the roots of $p^{\sigma_i}(x), i = 2, 3$ be y_1, y_2, y_3, y_4 so that $\sigma_i(c_0) = y_1 y_2 y_3 y_4$ and $\sigma_i(c_1) = -(y_1 y_2 y_3 y_4) \sum_{j=1}^4 (1/y_j)$. Let the y_j be ordered so that $B_i < y_4 < y_3 < y_2 < y_1 < 0$. So $\sigma_i(c_0) < (-y_1)(-B_i)^3$ and thus $(-1/y_1) < (-B_i)^3/\sigma_i(c_0)$. Also $\sigma_i(c_0) < (-y_2)^2(-B_i)^2$ so that $(-1/y_2) < ((-B_i)^2/\sigma_i(c_0))^{1/2}$. Continuing in this vein, we obtain

$$\sigma_i(c_1) < \sigma_i(c_0) \left[\frac{(-B_i)^3}{\sigma_i(c_0)} + \left(\frac{(-B_i)^2}{\sigma_i(c_0)} \right)^{1/2} + \left(\frac{-B_i}{\sigma_i(c_0)} \right)^{1/3} + \left(\frac{1}{\sigma_i(c_0)} \right)^{1/4} \right]. \quad (47)$$

In the other direction, from the arithmetic/geometric mean inequality, we deduce that

$$\sigma_i(c_1) \geq 4\sigma_i(c_0)^{3/4}. \quad (48)$$

In a similar way at the identity embedding, we obtain an upper bound on c_1 as

$$c_1 < c_0 \left[\left(\frac{G_u^2(-B_1)}{c_0} \right) + \left(\frac{G_u^2}{c_0} \right)^{1/2} \right] - 2R_\ell B_1^2. \quad (49)$$

On the other hand,

$$\frac{c_1}{c_0} = - \left(\frac{1}{x_3} + \frac{1}{x_4} \right) - \left(\frac{1}{\gamma} + \frac{1}{\bar{\gamma}} \right),$$

where the roots of $p(x)$ are $\gamma, \bar{\gamma}, x_3, x_4$. The first term here is greater than $-2/B_1$ and the second is greater than $-2/|\gamma|$. Jørgensen's Lemma states $|\gamma| + |\beta_1| \geq 1$ in a discrete non-elementary group, thus $|\gamma| > 2 \cos 2\pi/7 - 1$ so that

$$c_1 > c_0 \left(\frac{-2}{B_1} - \frac{2}{2 \cos 2\pi/7 - 1} \right). \quad (50)$$

In an entirely analogous manner, we can obtain similar bounds for $c_1^{(1)}$ and $c_1^{(2)}$ depending on each $c_0^{(1)}$ and $c_0^{(2)}$ in a linked triple.

Thus for $i = 2, 3$ and $j = 1, 2$ we have

$$4\sigma_i(c_0^{(j)})^{3/4} < \sigma_i(c_1^{(j)}) < \sigma_i(c_0^{(j)}) \left[\frac{(-\beta_i)^3}{\sigma_i(c_0^{(j)})} + \cdots + \left(\frac{1}{\sigma_i(c_0^{(j)})} \right)^{1/4} \right]. \quad (51)$$

Using the symmetry of γ_1, γ_2 , we obtain

$$\left(\frac{-2}{\beta_1} - \frac{\beta_1}{16} \right) c_0^{(1)} < c_1^{(1)} < c_0^{(1)} \left[\left(\frac{16(-\beta_1)}{c_0^{(1)}} \right) + \left(\frac{16}{c_0^{(1)}} \right)^{1/2} \right] + 8\beta_1^2. \quad (52)$$

If $p_2(x)$ has roots $\gamma_2, \bar{\gamma}_2, z_3, z_4$, then

$$c_1^{(2)} = -|\gamma_2|^2(z_3 + z_4) - (\gamma_2 + \bar{\gamma}_2)z_3z_4.$$

Using the AM/GM inequality and the fact that $-1 < \beta_1 < z_3, z_4 < 0$ we have

$$c_1^{(2)} \geq |\gamma_2|^2 2(z_3 z_4)^{1/2} - 2\Re(\gamma_2)z_3 z_4 > 2(z_3 z_4)(|\gamma_2|^2 - 2\Re(\gamma_2)) > -\beta_1^2/2. \quad (53)$$

Also

$$c_1^{(2)} < c_0^{(2)} \left[\left(\frac{16(-\beta_1)}{c_0^{(1)}} \right) + \left(\frac{16}{c_0^{(1)}} \right)^{1/2} \right] - \beta_1^3. \quad (54)$$

We now further exploit relation (11) to deduce that

$$-\beta_1 c_1 = c_0^{(2)} c_1^{(1)} + c_0^{(1)} c_1^{(2)}. \quad (55)$$

This gives upper bounds for $c_1^{(1)}$ and its conjugates which, in many cases, are an improvement on those obtained at (51) and (52) since

$$c_1^{(1)} \leq \frac{-\beta_1}{c_0^{(2)}}(\text{maximum value of } c_1) - \frac{c_0^{(1)}}{c_0^{(2)}}(\text{minimum value of } c_1^{(2)}). \quad (56)$$

In fact, in this (7, 7, 4) case, we do not enumerate the candidates for c_1 and $c_1^{(2)}$, but use the upper bounds for c_1 from (47) and (49) and the lower bounds for $c_1^{(2)}$ from (51) and (53) in (56). Thus using (51), (52) and (56), the Basic Method yields candidates for $c_1^{(1)}$ which depend on each linked pair $(c_0^{(1)}, c_0^{(2)})$ (We drop c_0). Note that, from (46)

$$\beta_1^2 \mid \beta_1 c_1^{(1)} + c_0^{(1)} - c_0^{(2)},$$

and we further reduce our list of candidates to satisfy this condition. The total number of triples $(c_1^{(1)}, c_0^{(1)}, c_0^{(2)})$ at this stage is 2071.

Again using the Basic Methods, we determine, independently of the foregoing calculations, the candidates for c_3 and $c_3^{(1)}$ (there are 452 and 187 respectively). Now

$$c_2^{(1)} = \frac{1}{2}(c_3 + \beta_1 c_3^{(1)} + c_3^{(1)2}). \quad (57)$$

The basic inequalities that $c_2^{(1)}$ and its conjugates must satisfy together with the fact that the second derivative of $p_1^{\sigma_i}(x)$, $i = 2, 3$ must have two real roots in the interval $(\beta_i, 0)$ gives inequalities relating $c_3^{(1)}$ and $c_2^{(1)}$ and hence involving c_3 and $c_3^{(1)}$. We thus determine all pairs $(c_3, c_3^{(1)})$ which are linked by these inequalities. Furthermore, since $2 \mid c_3 + \beta_1 c_3^{(1)} + c_3^{(1)2}$ we reduce the pairs to satisfy this divisibility condition. We then solve for $c_2^{(1)}$ using (57) and drop c_3 . There are 2218 resulting pairs $(c_3^{(1)}, c_2^{(1)})$.

We now relate these linked pairs $(c_3^{(1)}, c_2^{(1)})$ to the linked pairs $(c_0^{(1)}, c_0^{(2)})$ by inequalities. Once again using the AM/GM inequality yields $\sigma_i(c_3^{(1)}) \geq 4\sigma_i(c_0^{(1)})^{1/4}$ and $\sigma_i(c_2^{(1)}) \geq 6\sigma_i(c_0^{(1)})^{1/2}$ for $i = 2, 3$. Also $\sigma_i(c_2^{(1)}) < 3\sigma_i(c_0^{(1)})^{1/2} + 3\beta_i^2$ for $i = 2, 3$. A bit of manipulation using the AM/GM inequality also yields $c_2^{(1)} > 2\sqrt{3}c_0^{(1)1/2}$. We thus determine all 4-tuples which are linked by these inequalities. There are a total of 74570.

We now have a collection of 1051111 5-tuples $(c_3^{(1)}, c_2^{(1)}, c_1^{(1)}, c_0^{(1)}, c_0^{(2)})$ indexed by the linked pairs $(c_0^{(1)}, c_0^{(2)})$. They must satisfy equation (46). Implementing this gives 1934 4-tuples (we drop $c_0^{(2)}$). Then requiring that the

first derivative of $p_1^{\sigma_i}(x)$, $i = 2, 3$ has three roots in the interval $(\beta_i, 0)$ (see §7) yields a list of 746 polynomials. These and their conjugates can then be numerically solved and only 8 polynomials have the correct distribution of real roots. All these 8 turn out to be reducible and so we do not obtain any groups in this $(7, 7, 4)$ case.

Comments on the other five cases

Case $(8, 8, 5)$. This is tackled in a very similar manner to the preceding $(7, 7, 4)$ case. The main difference is that in this case, we use the inequalities (47) to (56) to enumerate the candidates for $(c_1, c_1^{(1)}, c_1^{(2)})$ depending on the linked triple $(c_0, c_0^{(1)}, c_0^{(2)})$ which satisfy equation (55). As in the preceding case, we then determine the pairs $(c_4^{(1)}, c_3^{(1)})$ and relate them by inequalities to the linked pair $(c_0^{(1)}, c_0^{(2)})$. In this case

$$\beta_1^5 + c_4^{(1)}\beta_1^4 + c_3^{(1)}\beta_1^3 + c_2^{(1)}\beta_1^2 + c_1^{(1)}\beta_1 + c_0^{(1)} = -c_0^{(2)} \quad (58)$$

$$5\beta_1^4 + 4c_4^{(1)}\beta_1^3 + 3c_3^{(1)}\beta_1^2 + 2c_2^{(1)}\beta_1 + c_1^{(1)} = c_1^{(2)} \quad (59)$$

from which we obtain

$$3\beta_1^5 + 2c_4^{(1)}\beta_1^4 + c_3^{(1)}\beta_1^3 - c_1^{(1)}\beta_1 - c_1^{(2)}\beta_1 - 2c_0^{(1)} - 2c_0^{(2)} = 0. \quad (60)$$

We now determine all 6-tuples $(c_4^{(1)}, c_3^{(1)}, c_1^{(1)}, c_1^{(2)}, c_0^{(1)}, c_0^{(2)})$ which satisfy (60) and use (58) to determine $c_2^{(1)}$ from the remaining coefficients. Now as for degree 3, we reduce our collection by the condition that, at the non-identity place, the degree 3 polynomial which is the second derivative of $p_1(x)$ has three real roots in the interval $(\beta_2, 0)$ where $\beta_2 = -(2 + \sqrt{2})$. This gives us 95 polynomials which can then be solved numerically and none have five real roots in the interval $(\beta_2, 0)$. So there are no groups in this case.

Case $(8, 8, 4)$. A simplified version of the above yields three polynomials with the correct numbers of real roots, but at the identity embedding, the real roots do not lie in the interval $(\beta_1, 0)$ where $\beta_1 = -(2 - \sqrt{2})$.

Case $(12, 12, 4)$. In this case, five polynomials have the correct distribution of roots, but for 3 of them, the real roots do not lie in the interval $(\beta_1, 0)$ at the identity embedding and for the other two, the resulting $\gamma(12, 12)$ value lies outside the contour $\Omega_{12,12}$.

Case $(6, 6, 5)$. An even more simplified version of the above method yields 31 polynomials with 3 real roots in the interval $(-1, 0)$. For all but one of them, the associated $\gamma(6, 6)$ lies outside the contour $\Omega_{6,6}$. Thus there is one candidate to be considered by geometric methods.

Case $(6, 6, 4)$. Using the same techniques as above, there are 70 polynomials with the correct distribution of roots and such that $\gamma(6, 6)$ lies inside the contour $\Omega_{6,6}$, all of these needing further examination by geometric methods.

9 Finite Co-volume

In §§5, 6, 7 and 8, we have outlined the methods we applied to all the triples (p, q, r) which appear on the *Aspiring List*. The result is a set of irreducible polynomials of degree r over L whose complex roots γ satisfy all four conditions (alternatively Lemma 2.4) of Theorem 2.2 and also the inequalities of §3, meaning they are not obviously of infinite volume.

This means that γ determines a group Γ generated by elements of orders p, q which is a subgroup of an arithmetic Kleinian group and hence discrete. From a specific value of γ we can compute the (normalized) matrices A and B which represent the generators f, g of Γ (see §3). The inequalities of §3 are derived from the geometric result that, if Γ is to be of finite co-volume then it cannot be a free product so that the isometric circles of g and g^{-1} cannot lie within the intersection of the isometric circles of f and f^{-1} .

Two further, but more complicated, geometric conditions (temporarily labelled *Free2* and *Free3* for use in the Examples below), necessary for Γ not to be a free product have been given in [34] in terms of the locations of the isometric circles of combinations of f, g . These simply consist of looking at the images of the isometric circles of one generator, say g under the transformation f and trying to piece together a fundamental domain from the intersection pattern. For instance, illustrated below, although the isometric circles of g do not lie in the region bounded between the isometric circles of f we have $f(I(g) \cup I(g^{-1})) \cap (I(g) \cup I(g^{-1})) = \emptyset$ and so if we look at the region (where we write $I(g)$ to mean the disk bounded by $I(g)$ etc)

$$I(f) \cap I(f^{-1}) \setminus (I(g) \cup I(g^{-1}) \cup f(I(g) \cup I(g^{-1})))$$

shaded below one can show without too much effort that this region lies within a fundamental domain for $\langle f, g \rangle$ on $\hat{\mathbb{C}}$ and in fact $\langle f, g \rangle$ is free on generators.

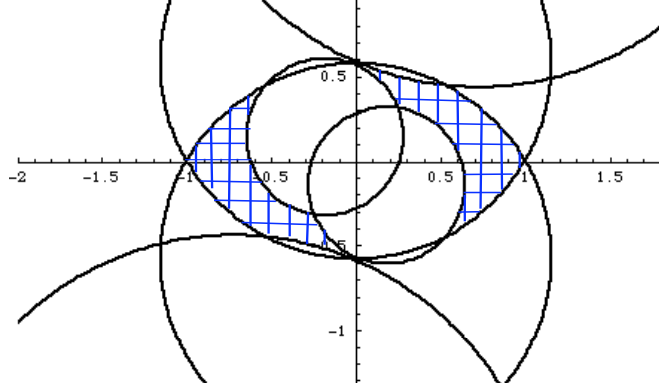


Diagram 3. Second level isometric circles

Of course one can go on looking at more and more isometric circles and their patterns and using this to formulate algebraic inequalities on γ . However after three levels this becomes quite impractical and we shall discuss below the computer program used to deal with these cases.

We apply these two elementary tests on γ to reduce the list of polynomials arising from §6,7 and 8.

Example 9.1 $(p, q, r) = (10, 10, 3)$. From Table 8, there are 5 candidates for γ which satisfy the four conditions of Theorem 2.2 and the inequalities of §3. Their minimum polynomials over $L = \mathbb{Q}(\sqrt{5})$ are given below.

No.	γ	polynomial
1	$-4.918226 + 5.698268i$	$x^3 + \frac{13+3\sqrt{5}}{2}x^2 + (30 + 12\sqrt{5})x + 1$
2	$0.635991 + 5.238279i$	$x^3 + (1 - \sqrt{5})x^2 + \frac{31+11\sqrt{5}}{2}x + 1$
3	$3.251943 + 8.478242i$	$x^3 + (-2 - 2\sqrt{5})x^2 + (42 + 18\sqrt{5})x + \frac{3+\sqrt{5}}{2}$
4	$8.794158 + 4.828433i$	$x^3 + \frac{-15-9\sqrt{5}}{2}x^2 + (51 + 22\sqrt{5})x + \frac{3+\sqrt{5}}{2}$
5	$6.180432 + 10.631111i$	$x^3 + \frac{-9-7\sqrt{5}}{2}x^2 + (77 + 33\sqrt{5})x + (3 + \sqrt{5})$

Test Free2 removes cases 1 and 5, while test Free3 removes case 2 and 3, leaving just one possibility to consider further.

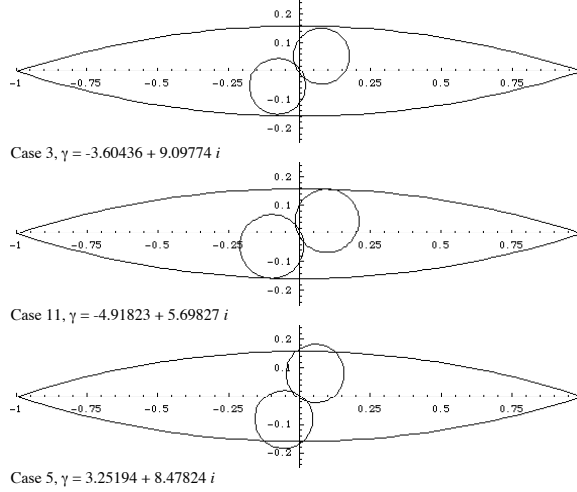


Diagram 4. $(10, 10, 3)$ -cases.

We list the numbers of candidates brought forward from §6,7 and 8 and in Table 9 show that many are eliminated using these Tests. Thus under “FT2”, “FT3”, are the number eliminated using FreeTest2 and FreeTest3 respectively. We also remove any reducible polynomials or duplicates which have survived to this stage. The final column gives the numbers of polynomials which are passed to the next step in procedure given below.

Finally, a computer program has been developed, initially by J. McKenzie, and named JSnap to study subgroups Γ of $\text{PSL}(2, \mathbb{C})$ which have two generators of finite order. This is effectively an implementation of the Dirichlet routine in J. Weeks’ program Snappea. This program aims to find a Dirichlet region for the group $\Gamma = \langle f, g \rangle$. A very important point to note here is that we know *a priori* that the group in question is discrete. However it is theoretically possible that the group Γ is geometrically infinite and so computationally impossible to identify a fundamental domain. JSnap runs and either produces a fundamental domain - either of finite or infinite volume - or produces an error message if it can’t put together a fundamental domain after looking at words of a given bounded length. In our situation JSnap always produces a fundamental domain which is either compact or meets the sphere at infinity in an open set (which itself will be a fundamental domain for the action of Γ on $\hat{\mathbb{C}} \setminus \Lambda(\Gamma)$). In this latter case the group cannot be of finite co-volume (and it might also not be free on generators - for instance certain Web-groups may arise) and so we can eliminate these cases.

Triple	No.	FT2	FT3	Rem.
(30, 30, 2)	1	0	0	1
(18, 18, 2)	2	1	1	1
(18, 9, 2)	2	2	0	0
(14, 7, 2)	1	0	1	0
(12, 12, 2)	18	7	4	7
(12, 6, 2)	1	0	1	0
(10, 10, 2)	15	8	2	5
(9, 9, 2)	2	1	1	0
(8, 8, 2)	20	8	4	8
(7, 7, 2)	8	4	2	2
(6, 6, 2)	24	9	5	10
(12, 12, 3)	19	10	5	4
(12, 6, 3)	1	1	0	0
(10, 10, 3)	5	2	2	1
(8, 8, 3)	29	13	10	5
(7, 7, 3)	1	0	1	0
(6, 6, 3)	124	52	30	38
(6, 6, 5)	1	0	0	1
(6, 6, 4)	70	36	19	15

Table 9: Geometric Test Results

If the fundamental domain found by JSnap is compact, then JSnap also returns an approximate co-volume.

In this way the remaining possibility for $(p, q, r) = (10, 10, 3)$ $\gamma = 8.794158 + 4.828433i$ is shown to have a fundamental domain which meets the sphere at infinity in an open set, and thus cannot be arithmetic.

Applying JSnap to the 15 cases in $(6, 6, 4)$ shows that they all have infinite volume and so there are no corresponding arithmetic Kleinian groups. It is a similar story for $r = 3$ except that here we meet our first example whose isometric circle configuration and γ value are illustrated below.

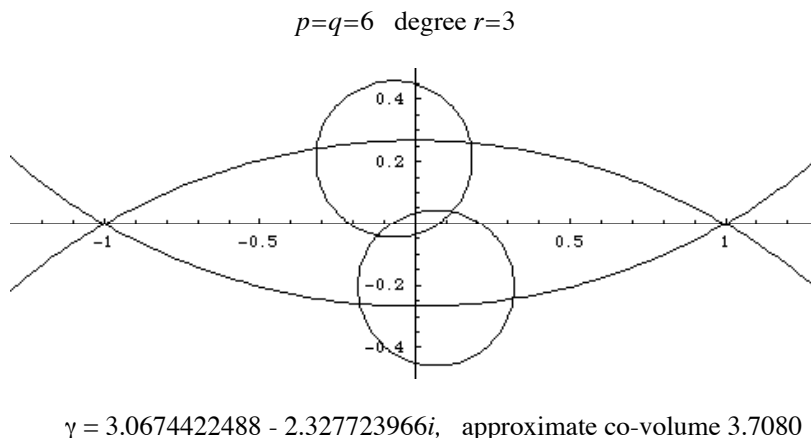


Diagram 5. A finite co-volume $(6, 6, 3)$ example.

10 The end product

After dealing with all the cases on the Aspiring List in the manner outlined in the proceeding sections we are left with just 17 complex values of γ whose corresponding group JSnap identifies as having finite co-volume and giving us an approximation to this co-volume. The groups corresponding to all but one of these values have already been identified as arithmetic in the literature in [25] as being obtained from surgery on 2-bridge knots and links. This can also be ascertained using Snappea as discussed in the introduction. The arithmetic data required to define these groups can be recovered from the polynomials satisfied by γ . The value $\sqrt{-3}$ gives a group which is not co-compact and has been discussed in [34]. In addition there is one further group, also not co-compact, which corresponds to the only real value of γ which arises for groups with generators of orders ≥ 6 (see [35, 36]).

In the tables below we list all the data on the groups we have found. The polynomial is that satisfied by $\gamma = \gamma(f, g)$ where $o(f) = p, o(g) = q$ over the field $\mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$. In most cases the description is given as an orbifold obtained by surgery on the one boundary component of a 2-bridge knot or on both boundary components of a 2-bridge link.

The appearance of the same description occurring twice in this table is discussed in the introduction as identification of different Nielsen classes of generators. In the other two cases, the description refers to [34].

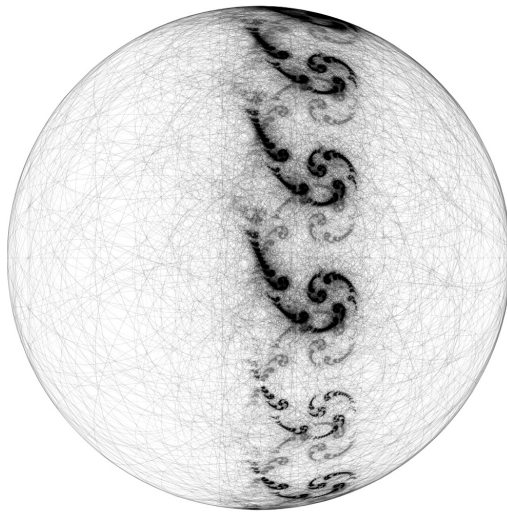
The commensurability class of the arithmetic group is, as we have discussed, determined by the field of definition and the defining quaternion algebra. In all the cases here, the discriminant Δ in the table uniquely describes the field given its degree and that it has one complex place. The quaternion algebra is determined by its ramification set which must include all real places so that the finite ramification suffices to identify the quaternion algebra. (The convention used here is that if a rational prime p splits as $\mathcal{P}_p\mathcal{P}'_p$ then these are ordered so that $N(\mathcal{P}_p) \leq N(\mathcal{P}'_p)$.) Note that the only commensurable pairs are those co-compact groups which are actually equal and given by different Nielsen classes of generators and the non-co-compact pair.

The orbifold volume is determined by Snappea and JSnap and the minimum volume is the smallest volume of any orbifold in the commensurability class which can be determined from the arithmetic data defining the commensurability class [3, 37].

No.	(p, q)	γ	poly
1	(12, 12)	$-0.259113 + 1.998874i$	$x^3 + (4 - 2\sqrt{3})x^2 + (11 - 4\sqrt{3})x + (7 - 4\sqrt{3})$
2	(12, 12)	$-0.633975 + 0.930605i$	$x^2 + (3 - \sqrt{3})x + (3 - \sqrt{3})$
3	(10, 10)	$-1 + 2.058171i$	$x^2 + 2x + (3 + \sqrt{5})$
4	(8, 8)	$-0.792893 + 0.978318i$	$x^2 + (3 - \sqrt{2})x + (3 - \sqrt{2})$
5	(6, 6)	$-1.877438 + 0.744861i$	$x^3 + 4x^2 + 5x + 1$
6	(6, 6)	$-2.884646 + 0.589742i$	$x^3 + 6x^2 + 10x + 2$
7	(6, 6)	$-0.891622 + 1.954093i$	$x^3 + 2x^2 + 5x + 1$
8	(6, 6)	$1.092519 + 2.052003i$	$x^3 - 2x^2 + 5x + 1$
9	(6, 6)	$3.067442 + 2.327724i$	$x^3 - 6x^2 + 14x + 2$
10	(6, 6)	$0.124046 + 2.836576i$	$x^3 + 8x + 2$
11	(6, 6)	$2.124407 + 2.746645i$	$x^3 - 4x^2 + 11x + 3$
12	(6, 6)	$4.109638 + 2.431700i$	$x^3 - 8x^2 + 21x + 5$
13	(6, 6)	$-1 + i$	$x^2 + 2x + 2$
14	(6, 6)	$-2 + 1.414214i$	$x^2 + 4x + 6$
15	(6, 6)	$1.732051i$	$x^2 + 3$
16	(6, 6)	$-1 + 2.645751i$	$x^2 + 2x + 8$
17	(6, 6)	$1 + 3i$	$x^2 - 2x + 10$
18	(6, 6)	-1	$x + 1$

No.	Description	Δ	$\text{Ram}_f(A)$	Orb. Vol.	Min. Vol.
1	$(12, 0), (12, 0)$ on $8/3$	-288576	\emptyset	3.3933	0.424167
2	$(12, 0)$ on $5/3$	-1728	$\mathcal{P}_2, \mathcal{P}_3$	1.8026	0.450658
3	$(10, 0)$ on $13/5$	-400	$\mathcal{P}_2, \mathcal{P}_5$	5.1674	1.291862
4	$(8, 0)$ on $5/3$	-448	$\mathcal{P}_2, \mathcal{P}_7$	1.5438	0.385966
5	$(6, 0)$ on $7/3$	-23	\mathcal{P}_3	2.0425	0.510633
6	$(6, 0), (6, 0)$ on $20/9$	-76	$\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}'_3$	5.2937	0.661715
7	$(6, 0), (6, 0)$ on $8/3$	-31	\mathcal{P}_3	2.6386	0.065965
8	$(6, 0)$ on $7/3$	-23	\mathcal{P}_3	2.0425	0.510633
9	$(6, 0)$ on $13/3$	-44	\mathcal{P}_2	3.7068	0.066194
10	$(6, 0)$ on $13/3$	-44	\mathcal{P}_2	3.7068	0.066194
11	$(6, 0)$ on $15/11$	-31	\mathcal{P}'_3	4.2217	0.263861
12	$(6, 0)$ on $65/51$	-23	\mathcal{P}_5	8.7986	0.078559
13	$(6, 0)$ on $5/3$	-4	$\mathcal{P}_2, \mathcal{P}_3$	1.2212	0.305322
14	$(6, 0), (6, 0)$ on $12/5$	-8	$\mathcal{P}_2, \mathcal{P}_3$	4.0153	0.250960
15	Non-compact Γ_{21}	-3	\emptyset	1.0149	0.253735
16	$(6, 0), (6, 0)$ on $30/11$	-7	$\mathcal{P}_2, \mathcal{P}_3$	7.1113	0.888915
17	$(6, 0), (6, 0)$ on $24/7$	-4	$\mathcal{P}_2, \mathcal{P}_5$	6.1064	0.152661
18	Non-compact Γ_{20}	-3	\emptyset	0.5074	0.084578

Remark. In eliminating certain candidates because they are not of co-finite volume - and which are guaranteed discrete by our arithmetic criteria - we ran into a number of interesting examples where our computational package JSnap had difficulty. This was largely to do with accumulation of roundoff error. In exploring these groups (looking for regions of discontinuity) we made use of the package “lim” developed by C. McMullen to draw the limit sets of Kleinian groups. The limit set of one such group is illustrated below. After seeing these pictures we were encouraged to modify our code to run on a different platform with higher precision to get an infinite volume fundamental region. However it is clear that in these sorts of cases (with parameters algebraic integers of low degree) that working with a version of Snap (the precise arithmetic version of Snappea developed by O. Goodman et al, [12]) would be the correct way forward. We are currently developing this program which will surely be necessary in extending our results beyond the cases $p, q \geq 6$.



Limit set of Kleinian group with two generators of order 12 and $\gamma = 2.73205 + 3.193141i$, a root of $x^2 - 2(1 + \sqrt{3})x + (9 + 5\sqrt{3}) = 0$

10.1 Generalised Triangle Groups

Here we prove Corollary 1.3. First we note that the only groups we need to consider here are the surgeries on two-bridge knot and link groups and thus the following lemma will suffice.

Lemma 10.1 *Let Γ be the orbifold fundamental group of $(p, 0)$ - $(q, 0)$ ($p, q \geq 2$) Dehn surgery on a two-bridge knot or link. Then Γ does not have a presentation as a generalised triangle group.*

Proof. Every element g of finite order in Γ has a nontrivial fixed point set in \mathbb{H}^3 which projects to an edge in the singular set of \mathbb{H}^3/Γ . Elements in the same conjugacy class project to the same edge. Next $(p, 0)$ - $(q, 0)$ Dehn surgery on a two-bridge knot or link has at most two components in its singular set (one if it is a knot). Let us denote the two primitive generators of order p and q arising from this surgery as f and g , so $\Gamma = \langle f, g \rangle$. Suppose Γ has a presentation of the form

$$\langle a^r = b^s = w(ab)^t \rangle, \quad r, s, t \geq 2 \quad (61)$$

There are at most two conjugacy classes of torsion in Γ . Thus a, b and any other element of finite order are conjugates of elements of $\langle f \rangle$ or $\langle g \rangle$. Thus, possibly increasing r or s and the complexity of w , we see that we can find a presentation of the form (61) with a and b conjugates of f and g and so $\{r, s\} \subset \{p, q\}$.

Suppose that b is not a conjugate of a . Then as w must also be conjugate into $\langle f \rangle$ or $\langle g \rangle$, the relation $w^r = 1$ is a direct consequence of the relators $a^p = b^q = 1$ and so $\Gamma = \langle a \rangle * \langle b \rangle$ which is not possible for a co-finite volume lattice. Thus b is a conjugate of a and $r = s$. This quickly implies that the abelianisation of Γ is a subgroup of $\langle a \rangle$ as w reduces to a power of a . Thus, if Γ has a presentation as at (61) we have deduced that Γ abelianises to a cyclic group \mathbb{Z}_k with $k|p$ or $k|q$. Further, we cannot be dealing with a knot surgery as a not conjugate to w implies two components to the singular set.

Next, every two bridge link has a presentation on a pair of meridians of the form $\langle u, v : uw = wu \rangle$ for w a word determined by the Schubert normal form [5]. Dehn surgery is equivalent to adding the relators $u^p = v^q = 1$ which then gives $\mathbb{Z}_p + \mathbb{Z}_q$ as the abelianisation.

Thus there can be no presentation as at (61) and the proof of the lemma is complete. \square

References

- [1] I. Agol, M. Belolipetsky, P. Storm and K. Whyte, *Finiteness of arithmetic hyperbolic reflection groups* arXiv:math.GT/0612132, v1, 5 Dec, 2006.
- [2] G. Baumslag, J. Morgan and P. Shalen, *Generalised triangle groups* Math. Proc. Camb. Phil. Soc., **102**, (1987), 25–31.

- [3] A. Borel *Commensurability classes and volumes of hyperbolic three-manifolds*, Ann. Sc. Norm. Pisa, **8**, (1981), 1 - 33.
- [4] B.H. Bowditch, C. Maclachlan and A.W. Reid, *Arithmetic hyperbolic surface bundles*. Math. Ann., **302**, (1995), 31–60
- [5] G. Burde and H. Zieschang, *Knots*, Studies in Maths. Vol 5. de Gruyter, Berlin (1985).
- [6] C. Cao and R. Meyerhoff, *The cusped hyperbolic 3-manifold of minimum volume*, Invent. Math., **146**, (2001), 451–478.
- [7] T. Chinburg and E. Friedman, *The smallest arithmetic hyperbolic three-orbifold*. Invent. Math., **86**, (1986), 507–527.
- [8] T. Chinburg, E. Friedman, K. Jones and A.W. Reid, *The arithmetic hyperbolic 3-manifold of smallest volume*, Ann. Scuola Norm. Sup. Pisa, **30**, (2001), 1–40.
- [9] H. Cohen, F. Diaz Y Diaz and M. Olivier, *Tables of octic fields with a quartic subfield*, Math. Comp., **68**, (1999), 1701–1716.
- [10] MDE Conder, GJ Martin, *Cusps, triangle groups and hyperbolic 3-folds* J. Austral. Math. Soc. Ser. A, **55**, (1993), 149–182.
- [11] M.D.E. Conder, C. Maclachlan, G.J. Martin and E.A. O’Brien, *2-generator arithmetic Kleinian groups III* Math. Scand., **90**, (2002), 161–179.
- [12] D. Coulson, O.A. Goodman, C.D. Hodgson and W.D. Neumann, *Computing arithmetic invariants of 3-manifolds*, Experiment. Math., **9**, (2000), 127–152.
- [13] F. Diaz Y Diaz, *Discriminant minimal et petis discriminants des corps de nombres de degre 7 avec cinq places reelles*, J. London Math. Soc., **38**, (1988), 33–46.
- [14] F. Diaz, Y Diaz and M. Olivier, *Corps imprimitifs de degre 9 de petit discriminant*, preprint.
- [15] B. Fine and G. Rosenberger, *A note on generalized triangle groups*, Abh. Math. Semj Univ. Hamburg, **56**, (1986), 233–244.

- [16] V. Flammang and G. Rhin, *Algebraic integers whose conjugates all lie in an ellipse*, Math. Comp. **74**, (2005), 2007–2014.
- [17] F. W. Gehring and G. J. Martin, *Stability and extremality in Jorgensen’s inequality*, Complex Variables, **12**, (1989), 277–282.
- [18] F. W. Gehring and G. J. Martin, *Commutators, collars and the geometry of Möbius groups*, J. d’Analyse Math., **63**, (1994), 175–219.
- [19] F. W. Gehring and G. J. Martin, *On the minimal volume hyperbolic 3-orbifold*, Math. Res. Letters, **1**, (1994), 107–114.
- [20] F. W. Gehring and G. J. Martin, *Minimal co-volume hyperbolic lattices. I. The spherical points of a Kleinian group*, Ann. of Math., **170**, (2009), 123–161.
- [21] F. W. Gehring, C. Maclachlan and G.J. Martin, *2-generator arithmetic Kleinian groups II*, Bull. Lond. Math. Soc., **30**, (1998), 258–266
- [22] F.W. Gehring, C. Maclachlan, G. Martin and A.W. Reid *Arithmeticity, Discreteness and Volume*, Trans. Amer. Math. Soc., **349**, (1997), 3611–3643.
- [23] M. Hagelberg, C. Maclachlan and G. Rosenberger *On discrete generalised triangle groups*, Proc. Edinburgh Math. Soc., **38**, (1995), 397 - 412.
- [24] H.M. Hilden, M.T. Lozano, and J.M. Montesinos-Amilibia, *The arithmeticity of certain torus bundle cone 3-manifolds and hyperbolic surface bundle 3-manifolds; and an enhanced arithmeticity test*, KNOTS ’96 (Tokyo), 73–80, World Sci. Publishing, River Edge, NJ, 1997.
- [25] H.M. Hilden, M.T. Lozano, and J.M. Montesinos-Amilibia, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant*, J. Knot Theory, Ramifications, **4**, (1995), 81–114.
- [26] H.M. Hilden, M.T. Lozano, and J.M. Montesinos-Amilibia, *A characterization of arithmetic subgroups of $SL(2, R)$ and $SL(2, C)$* , Math. Nachr., **159**, (1992), 245–270.
- [27] K.N. Jones, Kerry and A.W. Reid, *Minimal index torsion-free subgroups of Kleinian groups*. Math. Ann., **310**, (1998), 235–250.

- [28] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math., **98**, (1976), 739–749.
- [29] S. Katok, *Modular forms associated to closed geodesics and arithmetic applications*, Bull. Amer. Math. Soc., **11**, (1984), 177–179.
- [30] E. Klimenko, *Discrete groups in the 3-dimensional Lobachevskiĭspace that are generated by two rotations*, Siberian Math. J., **30** (1989), 95–100.
- [31] E. Klimenko and E. Kopteva, *All discrete \mathcal{RP} groups whose generators have real traces*, Internat. J. Algebra Comput., **15**, (2005), 577–618.
- [32] D. Long, C. Maclachlan and A.W. Reid, *Arithmetic Fuchsian groups of genus zero*, Pure and Applied Math. Quarterly, to appear.
- [33] C. Maclachlan and G.J. Martin, *On 2-generator Arithmetic Kleinian groups*. J. Reine Angew. Math., **511**, (1999), 95–117
- [34] C. Maclachlan and G.J. Martin, *The non-compact generalised arithmetic triangle groups*, Topology, **40**, (2001), 927–944.
- [35] C. Maclachlan and G. J. Martin, *All Kleinian groups with two elliptic generators whose commutator is elliptic*, Math. Proc. Cambridge Philos. Soc., **135**, (2003), 413–420.
- [36] C. Maclachlan and G. J. Martin, *The $(6,p)$ -arithmetic hyperbolic lattices in dimension 3*, Pure Appl. Math. Q., **7**, (2011), 365–382.
- [37] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Math., Springer–Verlag, 2003.
- [38] C. Maclachlan and A.W. Reid *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Camb. Phil. Soc., **102**, (1987), 251 - 258.
- [39] C. Maclachlan and A.W. Reid, *The arithmetic structure of tetrahedral groups of hyperbolic isometries*, Mathematika, **36**, (1989), 221–240.
- [40] C. Maclachlan and G. Rosenberger, *Two-generator arithmetic Fuchsian groups, II*, Math. Proc. Cambridge Philos. Soc., **111**, (1992), 7–24.

- [41] T.H. Marshall and G.J. Martin, *Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group*, Ann. of Math., **176**, (2012), 261–301.
- [42] B. Maskit, *Kleinian Groups*, Springer–Verlag, 1988.
- [43] R. Meyerhoff, *The cusped hyperbolic 3-orbifold of minimum volume*, Bull. Amer. Math. Soc., **13**, (1985), 154–156.
- [44] H.P. Mullholland, *The product of n complex homogeneous linear forms*, J. London Math. Soc., **35**, (1960), 241–250.
- [45] W.D. Neumann and A.W. Reid, *Arithmetic of hyperbolic manifolds*, Topology '90 (Columbus, OH, 1990), 273–310, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.
- [46] V.V. Nikulin, *Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces*, arXiv:math.AG/0609256, v1, 9 Sep, 2006.
- [47] A.M. Odlyzko, *Some analytic estimates of class numbers and discriminants*, Invent. Math., **29**, (1975), 275–286.
- [48] A.M. Odlyzko, *Discriminant Bounds* available from www.dtc.umn.edu/odlyzko/unpublished/discr.bound
- [49] A. W. Reid, *Arithmeticity of Knot Complements*, J. London Math. Soc., **43**, (1991), 171–184.
- [50] C.A. Rodgers, *The product of n real homogeneous linear forms*, Acta Math., **82**, (1950), 185–208.
- [51] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.
- [52] Z. Rudnick and P. Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys., **161**, (1994), 195–213.
- [53] P. Sarnak, *The arithmetic and geometry of some hyperbolic three-manifolds*, Acta Math, **151**, (1983), 253–295.

- [54] I. Schur, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*. Math. Zeit., **1**, (1918), 377 - 402.
- [55] H.M. Stark, *Some effective cases of the Brauer-Siegel Theorem*, Invent. Math., **23**, (1974), 135–152.
- [56] K. Takeuchi, *Arithmetic triangle groups*, J. Math. Soc. Japan, **29**, (1977), 91–106.
- [57] K. Takeuchi, *Arithmetic Fuchsian groups of signature $(1; e)$* , J. Math. Soc. Japan, **35**, (1983), 381–407.
- [58] W.P. Thurston, *Three-dimensional geometry and topology*, Vol. 1., Edited by Silvio Levy. Princeton Mathematical Series, **35**, Princeton University Press, Princeton, NJ, 1997.
- [59] W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc., **6**, (1982), 357–381.
- [60] M-F. Vigneras, *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Mathematics, No. 800, Springer-Verlag, 1980.
- [61] E.B. Vinberg, *Discrete linear groups that are generated by reflections*, Math. USSR Sbornik, **114**, (1967), 429–444.
- [62] E.B. Vinberg, *The absence of crystallographic groups of reflections in Lobachevski spaces of large dimension*, Trans. Moscow Math. Soc. **47** (1985), 75–112.
- [63] E.B. Vinberg, Ī. Mennike and Kh. Khelling, *On some generalized triangular groups and three-dimensional orbifolds*, Trudy Moskov. Mat. Obshch., **56**, (1995), 5–32, (Trans. Moscow Math. Soc., 1995, 1–21)
- [64] J. R. Weeks, *SnapPea: A computer program for creating and studying hyperbolic 3-manifolds*, available from <http://www.northnet.org/weeks/>, 2001.